A quasi-static stability analysis for Biot's equation and standard dissipative systems

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Abstract

In this paper, an extended version of Biot's differential equation is considered in order to discuss the quasi-static stability of a response for a solid in the framework of generalized standard materials. The same equation also holds for gradient theories since the gradients of arbitrary order of the state variables and of their rates can be introduced in the expression of the energy and of the dissipation potentials. The stability of a quasi-static response of a system governed by Biot's equations is discussed. Two approaches are considered, by direct estimates and by linearizations. The approach by direct estimates can be applied in visco-plasticity as well as in plasticity. A sufficient condition of stability is proposed and based upon the positivity of the second variation of energy along the considered response. This is an extension of the criterion of second variation, well known in elastic buckling, into the study of the stability of a response. The linearization approach is available only for smooth dissipation potentials, i.e. for t on asymptotic stability. The paper is illustrated by a simple example.

Keywords: Biot's equation; Local and non-local descriptions; Generalized standard models; Plasticity; Visco-plasticity; Stability of a quasi-static response; Second variation criterion

1. Introduction

In stability analysis, in particular in the theory of elastic buckling and of plastic buckling of solids, the characterization of the stability of an equilibrium is a well known and well developed subject, cf. for example the works of Koiter (1945) and Hill (1958). The criterion of second variation of energy and its counterpart in plasticity have been mathematically justified and applied with great success in various applications, cf. for example Nguyen (1994). In particular, the linearization method gives a strong tool to discuss the stability as well as the loss of stability of an equilibrium in elasticity and anelasticity.

The stability of the response of a solid submitted to a given loading path under initial perturbations represents the uniform continuity of the solution with respect to the initial conditions. This notion is stronger than the stability of an equilibrium position. It has been discussed by several authors in mathematics as well as in mechanics, cf. for example

Roseau (1966) or Hahn (1967), Petryk (1982), Nguyen (2000). The need for such an extension comes principally from the study of the stability of solids with time-dependent material behaviour such as visco-elastic or visco-plastic deformation. For example, the interpretation of the instability by localization, which can be observed in an experiment of uniaxial traction of a metallic specimen, requires a refined discussion on the growth of a perturbed solution from its homogeneous response. In visco-elasticity and visco-plasticity, the linearization method is very popular and has been considered by several authors for the stability analysis of evolving solutions, cf. for example Molinari and Clifton (1987), Nestorovic et al. (2000), Benallal and Comi (2003). In particular, it has been shown that most predictions by linearization are rather good compared to the observed experimental results.

It is well known that, from Lyapunov's theorem, the linearization method is fully justified in the stability analysis of an equilibrium position. However, it furnishes only some partial mathematical results on the stability analysis of an evolving solution since the linearized perturbed motions are then governed by a non-autonomous differential equation, cf. Roseau (1966), Hahn (1967) for example. Thus, for the stability analysis of a response of a solid, further discussions are still necessary.

The case of the generalized standard model (GSM) of solids is considered here in view of some complementary discussions on the stability analysis of an evolving solution. The description of GSM-models is based upon the assumptions of existence of an energy and a dissipation potential. The adopted model leads to Biot's differential equation and covers most usual laws of visco-plasticity and plasticity. From the assumption of existence of energy, this description includes the theory of elasticity as a particular case. It leads to the introduction of the criterion of second variation of energy in the same spirit as in the theory of elastic buckling. This criterion is here established for a quasi-static response of a visco-plastic or elasto-plastic solid by an approach based on some direct estimates of the perturbed problem. In visco-elasticity, the dissipation potential is smooth and the approach by linearization can be then considered. This approach leads to some sufficient conditions on the asymptotic stability of a quasi-static response and leads again to the criterion of second variation.

2. Biot's equation

2.1. Biot's equation and the GSM framework

In the rich and multifaceted works of M.A. Biot, the differential equation

$$J(q,\dot{q},\ddot{q}) + W_{,q}(q) + D_{,\dot{q}(q,\dot{q})} = F(q,\dot{q},t) \tag{1}$$

has been introduced and much discussed by Biot and co-workers, cf. Biot (1965), Biot (1977), in visco-elasticity. For a discrete system, $q \in R^n$ is a vector representing a set of n independent parameters, which usually represent the displacement and the internal state parameters, $W(q)$ is the energy potential and represents the recoverable energy stored in the system and $D(q,\dot{q})$ is the dissipation potential, in the spirit of Thomson in inviscid fluids. The generalized forces J and F are respectively the inertia and the external forces. In a purely mechanical description, q is reduced to the displacement, and Biot's equation results simply from the virtual work equation under the assumptions of internal forces admitting as reversible potential W and as dissipative potential D. If q_i is a displacement parameter, J_i is obtained from Lagrange formula and the associated equation is

$$J_i + W_{,q_i} + D_{,\dot{q}_i} = F_i, \qquad J_i = \frac{\mathrm{d}}{\mathrm{d}t} K_{,\dot{q}_i} - K_{,q_i} \tag{2}$$

where $K(q,\dot{q}) = \frac{1}{2}\dot{q} \cdot M(q) \cdot \dot{q}$ denotes the kinetic energy of the system. If q_i is an internal parameter, the associated forces J_i and F_i are null and the following relation holds

$$W_{,q_i} + D_{,\dot{q}_i} = 0. \tag{3}$$

In an equivalent way, for such a parameter, the driving force A_i defined by

$$A_i = -W_{,q_i} \tag{4}$$

satisfies also

$$A_i = D_{,\dot{q}_i}. \tag{5}$$

In other words, Eq. (3) also means the equilibrium condition for the reversible and dissipative internal forces, defined respectively from the energy potential and the dissipative potential. The fact that internal parameters may represent various physical phenomena and internal forces are often defined from reversible and dissipative potentials explains the great interests of Biot's equation in the study of physical systems. The last two equations are the governing equations for the model of generalized standard materials (GSM). In this framework, the general case of nonlinear potentials has been discussed in details for the constitutive modeling in visco-plasticity and in plasticity. In particular, the case of a dissipation potential homogeneous of degree 1 which is a convex but non-differentiable function has been the subject of several discussions, cf. Moreau (1970), Germain (1973), Halphen and Nguyen (1975), Lemaitre and Chaboche (1985), Nguyen (2000) etc. It has been shown that convex analysis gives the mathematical framework to extend Biot's equation for convex but non-differentiable potentials. In Eq. (1), the derivative $D_{,\dot{q}}$ must be understood in the sense of sub-gradient, cf. Moreau (1970), Rockafellar (1970), Frémond (2002). In particular, (5) is a differential equation relating the driving force A_i to the rate $\dot{q}_i$, denoted in the literature as the complementary equation.

2.2. Including the gradients

For a continuous system of state variable $\mathbf{q}$, which represents tensor fields defined in a volume Ω, of local value $q(x)$, $x \in \Omega$, and of energy and dissipation potentials

$$\mathbf{W}(\mathbf{q}) = \int_{\Omega} W \,\mathrm{d}V, \qquad \mathbf{D}(\dot{\mathbf{q}}) = \int_{\Omega} D \,\mathrm{d}V, \tag{6}$$

the associated Biot's equation is

$$\begin{cases} \delta\mathbf{W}(\mathbf{q}) + \delta\mathbf{D}(\dot{\mathbf{q}}) = \mathbf{F}\cdot\delta\mathbf{q} \quad \text{with} \\ \delta\mathbf{W}(\mathbf{q}) = \mathbf{W}_{,\mathbf{q}}\cdot\delta\mathbf{q},\, \delta\mathbf{D}(\dot{\mathbf{q}}) = \mathbf{D}_{,\dot{\mathbf{q}}}\cdot\delta\mathbf{q} \end{cases} \tag{7}$$

where $\mathbf{F}\cdot\delta\mathbf{q}$ denotes an appropriate linear form of $\delta\mathbf{q}$. It is of interests to give the local expressions of Biot's equation for a solid when the energy and dissipation potentials and the applied forces have the following form:

$$\begin{cases} \mathbf{W}(\mathbf{q}) = \int_{\Omega} W(q, \nabla q)\,\mathrm{d}V, \\ \mathbf{D}(\dot{\mathbf{q}}) = \int_{\Omega} D(\dot{q}, \nabla\dot{q})\,\mathrm{d}V, \\ \mathbf{F}\cdot\delta\mathbf{q} = \int_{\Omega} F\cdot\delta q\,\mathrm{d}V + \int_{\partial\Omega} f\cdot\delta q\,\mathrm{d}a. \end{cases} \tag{8}$$

It is straightforward that

$$\begin{cases} \delta\mathbf{W}(\mathbf{q}) = \int_{\Omega}(W_{,q} - \nabla\cdot W_{,\nabla q})\cdot\delta q\,\mathrm{d}V + \int_{\partial\Omega} n\cdot W_{,\nabla q}\cdot\delta q\,\mathrm{d}a, \\ \delta\mathbf{D}(\dot{\mathbf{q}}) = \int_{\Omega}(D_{,\dot{q}} - \nabla\cdot D_{,\nabla\dot{q}})\cdot\delta q\,\mathrm{d}V + \int_{\partial\Omega} n\cdot D_{,\nabla\dot{q}}\cdot\delta q\,\mathrm{d}a. \end{cases} \tag{9}$$

It follows that the equivalent local equations consist of:

- the body equations

$$W_{,q} + D_{,\dot{q}} - \nabla\cdot(W_{,\nabla q} + D_{,\nabla\dot{q}}) = F - \rho_o\ddot{u}, \quad \forall x \in \Omega; \tag{10}$$

- the boundary conditions

$$(W_{,\nabla q} + D_{,\nabla\dot{q}})\cdot n = f, \quad \forall x \in \partial\Omega. \tag{11}$$

The presence of higher gradients can be taken into account in the same spirit. For example, a second-gradient description is obtained if:

$$\begin{cases} \mathbf{W}(\mathbf{q}) = \int_{\Omega} W(q, \nabla q, \nabla\nabla q)\,\mathrm{d}V, \\ \mathbf{D}(\dot{\mathbf{q}}) = \int_{\Omega} D(\dot{q}, \nabla\dot{q}, \nabla\nabla\dot{q})\,\mathrm{d}V, \\ \mathbf{F}\cdot\delta\mathbf{q} = \int_{\Omega} F\cdot\delta q\,\mathrm{d}V + \int_{\partial\Omega} f\cdot\delta q\,\mathrm{d}a + \int_{\partial\Omega} \psi\cdot\nabla\delta q\,\mathrm{d}a. \end{cases} \tag{12}$$

In this case, since

$$\begin{cases} \delta\mathbf{W}(\mathbf{q}) = \int_{\Omega}(W_{,q} - \nabla\cdot W_{,\nabla q} + \nabla\cdot\nabla\cdot W_{,\nabla\nabla q})\cdot\delta q\,\mathrm{d}V \\ \qquad + \int_{\partial\Omega} n\cdot(W_{,\nabla q} - \nabla\cdot W_{,\nabla\nabla q})\cdot\delta q\,\mathrm{d}a + \int_{\partial\Omega} n\cdot W_{,\nabla\nabla q}\cdot\nabla\delta q\,\mathrm{d}a, \\ \delta\mathbf{D}(\dot{\mathbf{q}}) = \int_{\Omega}(D_{,\dot{q}} - \nabla\cdot D_{,\nabla\dot{q}} + \nabla\cdot\nabla\cdot D_{,\nabla\nabla\dot{q}})\cdot\delta q\,\mathrm{d}V \\ \qquad + \int_{\partial\Omega} n\cdot(D_{,\nabla\dot{q}} - \nabla\cdot D_{,\nabla\nabla\dot{q}})\cdot\delta q\,\mathrm{d}a + \int_{\partial\Omega} n\cdot D_{,\nabla\nabla\dot{q}}\cdot\nabla\delta q\,\mathrm{d}a, \end{cases} \tag{13}$$

it follows that the associated local equations are:

- the body equations

$$W_{,q} + D_{,\dot{q}} - \nabla \cdot (W_{,\nabla q} + D_{,\nabla \dot{q}}) + \nabla \cdot \nabla \cdot (W_{,\nabla\nabla q} + D_{,\nabla\nabla \dot{q}}) = F - \rho_o \ddot{u}, \quad \forall x \in \Omega, \tag{14}$$

- the boundary conditions

$$\begin{cases} (W_{,\nabla\nabla q} + D_{,\nabla\nabla \dot{q}}) \cdot n = \psi, \\ \big(W_{,\nabla q} + D_{,\nabla \dot{q}} - \nabla \cdot (W_{,\nabla\nabla q} + D_{,\nabla\nabla \dot{q}})\big) \cdot n = f, \quad \forall x \in \partial\Omega. \end{cases} \tag{15}$$

2.3. *Examples in the constitutive modeling of continua*

2.3.1. *Second gradient elasticity*

In this case, Ω is a reference configuration of a solid undergoing a finite transformation. In the model of second gradient elasticity, $q(x) = u(x)$ is the displacement and the energy per unit volume is $W(\nabla u, \nabla\nabla u)$ with

$$\sigma = W_{,\nabla u}, \qquad m = W_{,\nabla\nabla u}. \tag{16}$$

The generalized force m_{ijk} associated with $u_{i,jk}$ is a tensor of order 3, symmetric with respect to the two last indices. The dissipation potential is identically null since there is no dissipation. With the following expression of the external force

$$\mathbf{F} \cdot \delta\mathbf{u} = \int_{\Omega} F \cdot \delta u + \int_{\partial\Omega} (f \cdot \delta u + \varphi \cdot \delta u_{,n}) \tag{17}$$

the local equations (14) and (15) are reduced to:

$$\begin{cases} \nabla \cdot (\sigma - \nabla \cdot m) + F - \rho_o \ddot{u} = 0, \quad \forall x \in \Omega, \\ (\sigma - n \cdot m_{,n}) \cdot n + \cdots = f, \quad n \cdot m \cdot n = \varphi, \quad \forall x \in \partial\Omega. \end{cases} \tag{18}$$

The introduction of the second order gradient of the displacement is very classical in 2D-elasticity for the theory of plates and shells. In 3D-elasticity, it is also well known since the works of Toupin. In particular, it has been introduced to prevent sharp localizations and to interpret the thickness of the localized-band in solids. For example, the following expression of the energy has been suggested, cf. Landau and Lifshits (1966)

$$W = W_1(\nabla u, T) + W_2(\nabla\nabla u), \quad W_2 = \frac{1}{2}\kappa u_{i,jk} u_{i,jk}$$

where W_1 is a non-convex function of ∇u.

2.3.2. *Usual laws in visco-elasticity, visco-plasticity and plasticity*

In visco-elasticity, the dissipation potential $D(\dot{q})$ is a convex and smooth function of $\dot{q}$. In visco-plasticity and in plasticity, this potential is convex but not differentiable at the origin of fluxes. It is well known in visco-plasticity that the visco-plastic potential could be advantageously introduced by Legendre–Fenchel transform, cf. Fig. 1

$$D^*(A) = \max_{\dot{\alpha}} A \cdot \dot{\alpha} - D(\dot{\alpha}) \tag{19}$$

to write the complementary law under the form

$$\dot{\alpha} = D^*_{,A}(A) \tag{20}$$

for an internal parameter. For example, the Perzyna's model gives $D^*(A) = \frac{1}{2\eta}\langle \|A\| - k\rangle^2$ and the Norton–Hoff's model leads to $D^*(A) = \frac{1}{m\eta}\langle f(A)\rangle^m$ where $f(A) < 0$ denotes the elastic domain, f is a convex function such that $f(0) < 0$. If $\eta \Rightarrow 0$, an elastic-plastic model of elastic domain $f(A) \leqslant 0$ is obtained, with normality law

$$\dot{\alpha} = \mu \frac{\partial f}{\partial A}, \quad f \leqslant 0, \ \mu \geqslant 0, \ f\mu = 0. \tag{21}$$

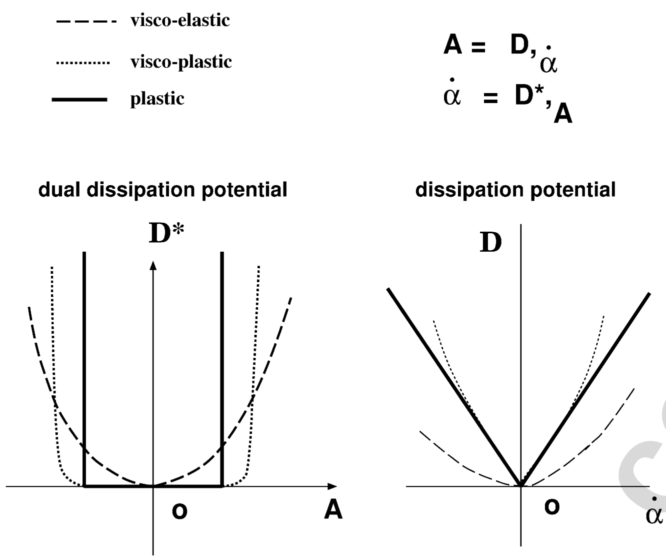

Fig. 1. The dissipation and dual-dissipation potentials in visco-elasticity, visco-plasticity and in plasticity.

2.3.3. Including the first gradient of the internal parameters

For a solid admitting as state variable $\mathbf{q} = (\mathbf{u}, \alpha)$, the introduction of the gradient of the internal parameter α, $\nabla\alpha$ in the expression of the energy potential and $\nabla\dot\alpha$ in the dissipation potential, leads from (10) and (11) to the local equations

$$W_{,\alpha} + D_{,\dot\alpha} - \nabla \cdot (W_{,\nabla\alpha} + D_{,\nabla\dot\alpha}) = F_\alpha, \quad \forall x \in \Omega, \tag{22}$$

$$(W_{,\nabla\alpha} + D_{,\nabla\dot\alpha}) \cdot n = f_\alpha, \quad \forall x \in \partial\Omega. \tag{23}$$

For an internal parameter without external action ($F = 0$, $f = 0$), the complementary relations are thus:

$$W_{,\alpha} + D_{,\dot\alpha} - \nabla \cdot (W_{,\nabla\alpha} + D_{,\nabla\dot\alpha}) = 0, \quad \forall x \in \Omega, \tag{24}$$

$$(W_{,\nabla\alpha} + D_{,\nabla\dot\alpha}) \cdot n = 0, \quad \forall x \in \partial\Omega. \tag{25}$$

The constitutive relation (24) and the boundary condition (25) as well as the thermodynamic background of gradient theories have been proposed by several authors, cf. for example Frémond (1985), Maugin (1990), Frémond and Nedjar (1996), Svedberg and Runesson (1997), Lorentz and Andrieux (2003), Nguyen and Andrieux (2005) and the references quoted in these papers. A common notation consists of writing by definition

$$\frac{\delta W}{\delta\alpha} = W_{,\alpha} - \nabla \cdot W_{,\nabla\alpha} \tag{26}$$

to write the constitutive relation (24) as

$$\frac{\delta W}{\delta\alpha} + \frac{\delta D}{\delta\dot\alpha} = 0 \quad \forall x \in \Omega. \tag{27}$$

For example, the phase-field model has been much discussed recently in the study of different phenomena in damage mechanics and in material sciences. In particular, it gives some interesting results on the modeling of phase change as well as in the study of damage and crack propagation, cf. for example Henry and Levine (2004), Karma et al. (2001). In this model, the internal parameter $0 \leqslant \alpha \leqslant 1$ is a parameter representing a undamaged proportion, such as the proportion of intact interatomic links. For the description of cracks in an elastic solid in the framework of damage mechanics, Lemaitre and Chaboche (1985) the energy is chosen from a standard two-minimum Ginsburg–Landau description:

$$\begin{cases} W(\nabla u, \alpha) = V(\alpha) + \frac{c}{2}\|\nabla\alpha\|^2 + g(\alpha)(W^o(\nabla u) - W^c), \\ V(\alpha) = \alpha^2(1-\alpha)^2, \qquad W^o(\nabla u) \quad \text{is an elastic energy}, \\ g(\alpha) \text{ increasing function satisfying } g(0) = g'(0) = g'(1) = 0,\ g(1) = 1, \end{cases}$$

$g(\alpha) = (4 - 3\alpha)\alpha^3$ in Karma et al. (2001), Henry and Levine (2004). The dissipation potential is simply $D(\dot{\alpha}) = \frac{\eta}{2}\dot{\alpha}^2$ and the governing local equations are

$$\begin{cases} \rho\ddot{u} - \nabla \cdot g(\alpha) W^o_{,\nabla u} = f_u, \\ \eta\dot{\alpha} + V'(\alpha) - c\Delta\alpha + g'(\alpha)(W^o(\nabla u) - W^c) = 0. \end{cases}$$

3. A discussion on quasi-static stability for standard dissipative systems

3.1. The governing equations

The case of a system governed by Biot's equation submitted to external forces $\mathbf{F}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{t})$ composed of a conservative force and a dissipative force admitting respectively as energy potential $\mathbf{P^c}(\mathbf{q}, \mathbf{t})$ and as dissipative potential $\mathbf{P^d}(\dot{\mathbf{q}}, \mathbf{t})$:

$$\begin{cases} \mathbf{F}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{t}) = \mathbf{F^c}(\mathbf{q}, \mathbf{t}) + \mathbf{F^d}(\dot{\mathbf{q}}, \mathbf{t}), \\ \mathbf{F^c}(\mathbf{q}, \mathbf{t}) = -\mathbf{P^c_{,q}}(\mathbf{q}, \mathbf{t}), \qquad \mathbf{F^d}(\dot{\mathbf{q}}, \mathbf{t}) = -\mathbf{P^d_{,\dot{q}}}(\dot{\mathbf{q}}, \mathbf{t}) \end{cases} \tag{28}$$

is considered here for stability analysis. With the following notation

$$\overline{\mathbf{W}}(\mathbf{q}, \mathbf{t}) = \mathbf{W}(\mathbf{q}, \mathbf{t}) + \mathbf{P^c}(\mathbf{q}, \mathbf{t}), \qquad \overline{\mathbf{D}}(\dot{\mathbf{q}}, \mathbf{t}) = \mathbf{D}(\dot{\mathbf{q}}, \mathbf{t}) + \mathbf{P^d}(\dot{\mathbf{q}}, \mathbf{t}) \tag{29}$$

where $\overline{\mathbf{W}}(\mathbf{q}, \mathbf{t})$ and $\overline{\mathbf{D}}(\dot{\mathbf{q}}, \mathbf{t})$ denote the total energy potential and the total dissipation potential of the system, Biot's equation can be written under the form

$$\mathbf{J} + \overline{\mathbf{W}}_{,\mathbf{q}}(\mathbf{q}, \mathbf{t}) + \overline{\mathbf{D}}_{,\dot{\mathbf{q}}}(\dot{\mathbf{q}}, \mathbf{t}) = 0 \tag{30}$$

which is by definition the governing equation of standard dissipative systems. For simplicity, the dependence on t of the loading history is often described by a load parameter $\lambda(t)$ which is a given function. In this case, the total potentials are conveniently written as

$$\overline{\mathbf{W}} = \overline{\mathbf{W}}(\mathbf{q}, \lambda), \quad \overline{\mathbf{D}} = \overline{\mathbf{D}}(\dot{\mathbf{q}}, \lambda), \quad \lambda = \lambda(t). \tag{31}$$

Some general results concerning the quasi-static stability of the response of a standard dissipative system are discussed in the context of visco-elasticity, visco-plasticity and plasticity. With or without the presence of gradients in the modeling of the constitutive equations, the governing equations for a solid deal with the fields of displacement and internal parameters $\mathbf{q} = (\mathbf{u}, \alpha)$. In terms of the displacement $\mathbf{u}$ and internal parameter α, the governing equations are

$$\begin{cases} \mathbf{J} + \overline{\mathbf{W}}_{,\mathbf{u}}(\mathbf{u}, \alpha, \lambda) + \overline{\mathbf{D}}_{,\dot{\mathbf{u}}}(\dot{\mathbf{u}}, \dot{\alpha}, \lambda) = \mathbf{0}, \\ \overline{\mathbf{W}}_{,\alpha}(\mathbf{u}, \alpha, \lambda) + \overline{\mathbf{D}}_{,\dot{\alpha}}(\dot{\mathbf{u}}, \dot{\alpha}, \lambda) = \mathbf{0}, \\ \lambda = \lambda(t), \quad 0 \leqslant t < +\infty. \end{cases} \tag{32}$$

When u is not a dissipative mechanism as in classical plasticity and visco-plasticity, the governing equations are

$$\begin{cases} \mathbf{J} + \overline{\mathbf{W}}_{,\mathbf{u}}(\mathbf{u}, \alpha, \lambda) = \mathbf{0}, \\ \overline{\mathbf{W}}_{,\alpha}(\mathbf{u}, \alpha, \lambda) + \overline{\mathbf{D}}_{,\dot{\alpha}}(\dot{\alpha}, \lambda) = \mathbf{0}, \\ \lambda = \lambda(t), \quad 0 \leqslant t < +\infty. \end{cases} \tag{33}$$

Starting at time 0 from a given initial state of displacement and internal parameter $\mathbf{q}(0) = (\mathbf{u}(0), \alpha(0))$ and $\dot{\mathbf{u}}(0)$, the response of the solid is $\mathbf{q}(t)$, $t \in [0, +\infty[$. It is assumed that a bounded solution $\mathbf{q_o}(\mathbf{t})$ exists.

3.2. An approach by direct estimates

3.2.1. Assumptions

To avoid some mathematical difficulties on the choice of functional spaces for continua, only the discrete case obtained after a space discretization by f.e.m. of the previous governing equations for a solid Ω is considered although the proposed analysis could be extended in the same spirit. The following conditions are assumed:

(i) a state-independent and convex potential $\mathbf{D}$. Thus, in visco-plasticity and in plasticity, the elastic domain $\mathbf{C}$ is a state-independent convex domain containing the origin strictly in its interior,

(ii) the energy potential $\overline{\mathbf{W}}(\mathbf{q}, \lambda)$ is a smooth C^3-function, thus:

$$\begin{cases} |\overline{\mathbf{W}}_{,\lambda}(\mathbf{q}, \lambda)| \leqslant K_{01}, \\ |\overline{\mathbf{W}}_{,\mathbf{qq}}(\mathbf{q}, \lambda)[\tilde{\mathbf{q}}, \mathbf{q}^*]| \leqslant K_{20}\|\tilde{\mathbf{q}}\|\|\mathbf{q}^*\|, \\ |\overline{\mathbf{W}}_{,\mathbf{q}\lambda}(\mathbf{q}, \lambda)[\mathbf{q}^*]| \leqslant K_{11}\|q^*\|, \\ |\overline{\mathbf{W}}_{,\mathbf{qqq}}(\mathbf{q}, \lambda)[\tilde{\mathbf{q}}, \hat{\mathbf{q}}, \mathbf{q}^*]| \leqslant K_{30}\|\tilde{\mathbf{q}}\|\|\hat{\mathbf{q}}\|\|\mathbf{q}^*\|, \\ |\overline{\mathbf{W}}_{,\mathbf{qq}\lambda}(\mathbf{q}, \lambda)[\tilde{\mathbf{q}}, \hat{\mathbf{q}}]| \leqslant K_{21}\|\tilde{\mathbf{q}}\|\|\hat{\mathbf{q}}\| \end{cases} \tag{34}$$

for $\|\mathbf{q} - \mathbf{q_o}(\mathbf{t})\| \leqslant M$, $0 \leqslant \lambda \leqslant \lambda_M$.

(iii) the energy potential $\overline{\mathbf{W}}(\mathbf{q}, \lambda)$ is strongly convex in the sense that

$$\overline{\mathbf{W}}_{,\mathbf{qq}}(\mathbf{q}, \lambda)[\mathbf{q}^*, \mathbf{q}^*] \geqslant k_{20}\|\mathbf{q}^*\|^2, \tag{35}$$

where the coefficients k_{ij} are strictly positive constants, for $\|\mathbf{q} - \mathbf{q_o}(\mathbf{t})\| \leqslant M$, $0 \leqslant \lambda \leqslant \lambda_M$.

(iv) It is also assumed that the loading path, defined by the function $\lambda(t)$ on the interval $[0, \infty[$, satisfies

$$\int_0^{+\infty} \left|\dot{\lambda}(t)\right| \mathrm{d}t < +\infty \quad \text{i.e. } \dot{\lambda} \in L^1(0, \infty). \tag{36}$$

For example, this assumption is satisfied for a monotone loading history in $[0, \lambda_M]$.

3.2.2. Continuity of a response with respect to an initial disturbance

A quasi-static transformation is described by the governing equations

$$\begin{cases} \overline{\mathbf{W}}_{,\mathbf{u}}(\mathbf{u}, \alpha, \lambda) = \mathbf{0}, \\ \overline{\mathbf{W}}_{,\alpha}(\mathbf{u}, \alpha, \lambda) + \overline{\mathbf{D}}_{,\dot{\alpha}}(\dot{\alpha}, \lambda) = \mathbf{0}, \\ \alpha(0) = \alpha_o, \quad \lambda = \lambda(t), \quad 0 \leqslant t < +\infty. \end{cases} \tag{37}$$

Let $\mathbf{q_o}(\mathbf{t})$ be a bounded solution of this equation and $\mathbf{q}(\mathbf{t})$ be a perturbed solution associated with a different initial condition. The discussion consists of estimating the distance $\|\mathbf{q}(\mathbf{t}) - \mathbf{q_o}(\mathbf{t})\|$ for all t in terms of the initial distance in order to show that this distance is small if the initial distance is sufficiently small for stability analysis.

As in the classical proof of Lyapunov's theorem, the method consists of assuming first that $\|\mathbf{q} - \mathbf{q_o}(\mathbf{t})\| \leqslant M$ for all t in order to take the advantage of the introduced assumptions. A better estimate of this distance is then derived and justifies this working assumption.

A bounded solution $\mathbf{q}(\mathbf{t})$ satisfies some *a priori* estimates. After a multiplication by $\dot{\mathbf{q}}$, the governing equation leads to the energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{W}} + \mathbf{A} \cdot \dot{\alpha} = \overline{\mathbf{W}}_{,\lambda}\dot{\lambda}.$$

From the assumption on the elastic domain, it follows that

$$r\int_0^t \left\|\dot{\alpha}(s)\right\| \mathrm{d}s \leqslant \int_0^t \mathbf{A} \cdot \dot{\alpha}\, \mathrm{d}s = \overline{\mathbf{W}}_0 - \overline{\mathbf{W}}_t + K_{01}\int_0^t |\dot{\lambda}|\, \mathrm{d}s.$$

The energy $\overline{\mathbf{W}}_t$ remains bounded for all t since $\mathbf{q}$ is bounded by assumption, thus

$$\int_0^{+\infty} \|\dot{\alpha}\|\, \mathrm{d}t < +\infty.$$

The same conclusion concerning $\dot{\mathbf{u}}$ is also available. Indeed, the equilibrium equation $\overline{\mathbf{W}}_{,\mathbf{u}} \cdot \delta\mathbf{u} = 0$ gives after time differentiation

$$\overline{\mathbf{W}}_{,\mathbf{uu}}[\dot{\mathbf{u}}, \delta\mathbf{u}] + \delta\mathbf{u} \cdot \overline{\mathbf{W}}_{,\mathbf{u}\alpha} \cdot \dot{\alpha} + \dot{\lambda}\overline{\mathbf{W}}_{,\mathbf{u}\lambda} \cdot \delta\mathbf{u} = 0$$

and leads to $k_{20}\|\dot{\mathbf{u}}\| \leqslant K_{20}\|\dot{\alpha}\| + K_{11}|\dot{\lambda}|$. The rates $\dot{\mathbf{q}}_{\mathrm{o}}$ and $\dot{\mathbf{q}}$ thus belong to $L^1(0, +\infty, R^n)$.

On the other hand, the governing equation, written for solutions $\mathbf{q_o}$ and $\mathbf{q}$, gives after a combination of the obtained results

$$\left(\overline{\mathbf{W}}_{,\mathbf{q}} - \overline{\mathbf{W}}^{\mathbf{o}}_{,\mathbf{q}}\right) \cdot (\dot{\mathbf{q}} - \dot{\mathbf{q}}_{\mathbf{o}}) \leqslant 0 \quad \forall t.$$

Since

$$\begin{aligned}
&\left(\overline{\mathbf{W}}_{,\mathbf{q}} - \overline{\mathbf{W}}^{\mathbf{o}}_{,\mathbf{q}}\right) \cdot \mathbf{\Delta}\dot{\mathbf{q}} = \overline{\mathbf{W}}^{\mathbf{o}}_{,\mathbf{qq}}[\mathbf{\Delta q}, \mathbf{\Delta}\dot{\mathbf{q}}] + r_1, \\
&|r_1| \leqslant K_{30}\|\mathbf{\Delta q}\|^2\|\mathbf{\Delta}\dot{\mathbf{q}}\|, \\
&\overline{\mathbf{W}}^{\mathbf{o}}_{,\mathbf{qq}}[\mathbf{\Delta q}, \mathbf{\Delta}\dot{\mathbf{q}}] = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\overline{\mathbf{W}}^{\mathbf{o}}_{,\mathbf{qq}}[\mathbf{\Delta q}, \mathbf{\Delta q}]\right) - r_2, \\
&|r_2| \leqslant \frac{1}{2}\|\mathbf{\Delta q}\|^2\left(K_{30}\|\dot{\mathbf{q}}_{\mathbf{o}}\| + K_{21}|\dot{\lambda}|\right).
\end{aligned}$$

It follows finally that the quantity $h = \frac{1}{2}\overline{\mathbf{W}}^{\mathbf{o}}_{,\mathbf{qq}}[\mathbf{\Delta q}, \mathbf{\Delta q}]$ satisfies

$$\dot{h} \leqslant m(t)h(t), \quad m(t) = \frac{L}{k_{20}}\left(3\|\dot{\mathbf{q}}_{\mathbf{o}}\| + 2\|\dot{\mathbf{q}}\| + \left|\dot{\lambda}(t)\right|\right)$$

with $L = \max(K_{30}, K_{21})$. Thus, from Gronwall's lemma, the inequality

$$h(t) \leqslant h(0)\exp Z(t), \quad Z(t) = \int_0^t m(s)\,\mathrm{d}s \leqslant G$$

holds, where the constant G exists since $\dot{\mathbf{q}} \in L^1(0, +\infty, R^n)$, $\dot{\mathbf{q}}_{\mathbf{o}} \in L^1(0, +\infty, R^n)$ and $\dot{\lambda} \in L^1(0, +\infty)$ and leads to

$$\left\|\mathbf{\Delta q(t)}\right\|^2 \leqslant \frac{K_{20}\exp G}{k_{20}}\left\|\mathbf{\Delta q(0)}\right\|^2.$$

The stability of a bounded quasi-static solution $\mathbf{q_o(t)}$ is thus ensured under the considered assumptions.

3.3. The stability criterion of second variation of energy

In this proof, the local convexity of the energy potential $\overline{\mathbf{W}}$ is essential and leads to the criterion of second variation of energy

$$\delta^2\overline{\mathbf{W}} = \overline{\mathbf{W}}^{\mathbf{o}}_{,\mathbf{qq}}[\delta\mathbf{q}, \delta\mathbf{q}] > 0 \quad \forall\delta\mathbf{q} \neq \mathbf{0} \text{ and } \forall t > 0 \tag{38}$$

which ensures the stability of a response $\mathbf{q_o(t)}$ under the assumptions of convexity of the dissipation potential and the smoothness of energy. In fact, since $A - A_o$ remains small in the previous proof, the same conclusion also holds under the positivity of the second variation of energy for all

$$\delta\mathbf{q} = (\delta\mathbf{u}, \delta\alpha) \quad \textbf{admissible} \text{ at time } t \text{ (i.e. } \delta\alpha = \mathbf{0} \quad \forall x \in \Omega^{el}_o(t)) \tag{39}$$

where $\Omega^{el}_o(t)$ denotes the elastic zone at time t of $\mathbf{q_o}(t)$.

The criterion (38) also holds for the stability analysis of an equilibrium position which is simply a particular response to a constant load. In this case, it may be interesting to compare this criterion with Hill's stability criterion of an elastic-plastic equilibrium for the same solid. The criterion of second variation of energy is more conservative than Hill's criterion which requires the positivity of the same matrix on a smaller set $\delta\mathbf{q} = (\delta\mathbf{u}, \delta\alpha)$ with $\delta\alpha = \mu\mathbf{f}_{,\mathbf{A}}$ when the plastic criterion is given by the inequality $f(A) \leqslant 0$. The two criteria are however identical if the admissible sets for $\delta\alpha$ are identical, for example in the example of Shanley's column of the last section.

3.4. The linearization approach: a result of asymptotic stability

When the dissipation potential $\overline{\mathbf{D}}$ is a smooth function, i.e. essentially in visco-elasticity, the method of linearization can be introduced.

For a quasi-static evolution, the linearized equations near a regular solution $\mathbf{q_o(t)}$ are

$$\overline{\mathbf{W}}^{\mathbf{o}}_{,\mathbf{qq}} \cdot \mathbf{q}^* + \overline{\mathbf{D}}^{\mathbf{o}}_{,\dot{\mathbf{q}}\dot{\mathbf{q}}} \cdot \dot{\mathbf{q}}^* = \mathbf{0}, \quad \forall t > 0. \tag{40}$$

When the dissipation potential is strictly convex, the operator $\overline{\mathbf{D}}^{\mathbf{o}}_{,\dot{\mathbf{q}}\dot{\mathbf{q}}}$ is invertible and the linearized equations can also be written under the form

$$\frac{\mathrm{d}\mathbf{q}^*}{\mathrm{d}t} = \boldsymbol{\Psi}(\mathbf{t}) \cdot \mathbf{q}^* \quad \text{with } \boldsymbol{\Psi}(\mathbf{t}) = -\overline{\mathbf{D}}^{\mathbf{o}\,-\mathbf{1}}_{,\dot{\mathbf{q}}\dot{\mathbf{q}}} \overline{\mathbf{W}}^{\mathbf{o}}_{,\mathbf{qq}}, \tag{41}$$

which is a non-autonomous differential equation, cf. Abed Meraim (1999a), Abed Meraim (1999b), Hahn (1967) or Roseau (1966). The following statement holds:

$$\begin{cases} \text{If the bilinear forms } \overline{\mathbf{D}}^{\mathbf{o}}_{,\dot{\mathbf{q}}\dot{\mathbf{q}}} \text{ and } \overline{\mathbf{W}}^{\mathbf{o}}_{,\mathbf{qq}} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\overline{\mathbf{D}}^{\mathbf{o}}_{,\dot{\mathbf{q}}\dot{\mathbf{q}}}) \\ \text{are uniformly positive-definite } \forall t > 0, \\ \text{then the solution } \mathbf{q_o(t)} \text{ is asymptoticaly stable.} \end{cases} \tag{42}$$

The proof of this statement is straightforward for the linearized equation by taking as Lyapunov's functional $\boldsymbol{\Lambda}(\mathbf{t}) = \overline{\mathbf{D}}^{\mathbf{o}}_{,\dot{\mathbf{q}}\dot{\mathbf{q}}}(\mathbf{t})[\mathbf{q}^*(\mathbf{t}), \mathbf{q}^*(\mathbf{t})]$. This property still holds for the nonlinear equation.

In the particular case of an equilibrium $\mathbf{q_o(t)} = \mathbf{q_{eq}}\ \forall t$, it is well known that the classical Lyapunov's theorem gives a stronger statement concerning the occurrence of instability. This theorem is not recovered in the given proposition which is thus not optimal. However, without additional assumptions, no stronger statement can be derived. In particular, if the real parts of some eigenvalues of $\boldsymbol{\Psi}(\mathbf{t})$ are positive, no general conclusion is available.

If the dissipation potential depends only on $\dot{\alpha}$, the operator $\overline{\mathbf{D}}^{\mathbf{o}}_{,\dot{\mathbf{q}}\dot{\mathbf{q}}}$ is not invertible. However, the linearized equations (40) can be then explicitly written as

$$\begin{cases} \overline{\mathbf{W}}^{\mathbf{o}}_{,\mathbf{uu}} \cdot \mathbf{u}^* + \overline{\mathbf{W}}^{\mathbf{o}}_{,\mathbf{u}\alpha} \cdot \alpha^* = \mathbf{0}, \\ \overline{\mathbf{W}}^{\mathbf{o}}_{,\alpha\mathbf{u}} \cdot \mathbf{u}^* + \overline{\mathbf{W}}^{\mathbf{o}}_{,\alpha\alpha} \cdot \alpha^* + \overline{\mathbf{D}}^{\mathbf{o}}_{,\dot{\alpha}\dot{\alpha}} \cdot \dot{\alpha}^* = \mathbf{0} \end{cases} \tag{43}$$

which gives when the operator $\overline{\mathbf{W}}^{\mathbf{o}}_{,\mathbf{uu}}$ is invertible:

$$\begin{cases} \mathbf{u}^* = -\overline{\mathbf{W}}^{\mathbf{o}\,-\mathbf{1}}_{,\mathbf{uu}} \overline{\mathbf{W}}^{\mathbf{o}}_{,\mathbf{u}\alpha} \cdot \alpha^*, \\ (\overline{\mathbf{W}}^{\mathbf{o}}_{,\alpha\alpha} - \overline{\mathbf{W}}^{\mathbf{o}}_{,\alpha\mathbf{u}} \overline{\mathbf{W}}^{\mathbf{o}\,-\mathbf{1}}_{,\mathbf{uu}} \overline{\mathbf{W}}^{\mathbf{o}}_{,\mathbf{u}\alpha}) \cdot \alpha^* + \overline{\mathbf{D}}^{\mathbf{o}}_{,\dot{\alpha}\dot{\alpha}} \cdot \dot{\alpha}^* = \mathbf{0}. \end{cases} \tag{44}$$

The result (42) must be replaced by

$$\begin{cases} \text{If the bilinear forms } \overline{\mathbf{D}}^{\mathbf{o}}_{,\dot{\alpha}\dot{\alpha}} \text{ and } \overline{\mathbf{W}}^{\mathbf{o}}_{,\mathbf{uu}} \text{ and } (\overline{\mathbf{W}}^{\mathbf{o}}_{,\alpha\alpha} - \overline{\mathbf{W}}^{\mathbf{o}}_{,\alpha\mathbf{u}} \overline{\mathbf{W}}^{\mathbf{o}\,-\mathbf{1}}_{,\mathbf{uu}} \overline{\mathbf{W}}^{\mathbf{o}}_{,\mathbf{u}\alpha}) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\overline{\mathbf{D}}^{\mathbf{o}}_{,\dot{\alpha}\dot{\alpha}}) \\ \text{are uniformly positive-definite } \forall t > 0, \text{ then the solution } \mathbf{q_o(t)} \text{ is asymptoticaly stable.} \end{cases} \tag{45}$$

In particular, if the dissipation potential is a quadratic function, the results (42) and (45) are reduced to the criterion of second variation of energy (38).

For a dynamic response, the linearized equations are

$$\begin{cases} \mathbf{M}^{\mathbf{o}} \cdot \ddot{\mathbf{u}}^* + \overline{\mathbf{W}}^{\mathbf{o}}_{,\mathbf{uu}} \cdot \mathbf{u}^* + \overline{\mathbf{W}}^{\mathbf{o}}_{,\mathbf{u}\alpha} \cdot \alpha^* = \mathbf{0}, \\ \overline{\mathbf{W}}^{\mathbf{o}}_{,\alpha\mathbf{u}} \cdot \mathbf{u}^* + \overline{\mathbf{W}}^{\mathbf{o}}_{,\alpha\alpha} \cdot \alpha^* + \overline{\mathbf{D}}^{\mathbf{o}}_{,\dot{\alpha}\dot{\alpha}} \cdot \dot{\alpha}^* = \mathbf{0}. \end{cases} \tag{46}$$

The stability analysis for a dynamic response $\mathbf{q_o(t)}$ is not straightforward when $\overline{\mathbf{W}}^{\mathbf{o}}_{,\mathbf{qq}}$ and $\overline{\mathbf{D}}^{\mathbf{o}}_{,\dot{\alpha}\dot{\alpha}}$ are time-dependent.

4. A simple example: the Shanley's column

The discrete Shanley's column is considered here as an illustrating example in visco-elasticity, visco-plasticity or plasticity. For example, in visco-elasticity, the bars AE and BF are assumed to be visco-elastic following the Maxwell rheological model, cf. Fig. 2. Let $u = (z, \theta)$ be displacement parameters, $\alpha = (\alpha_1, \alpha_2)$ the internal parameters. Submitted to a vertical force of amplitude λ, the column under load is a standard dissipative system of energy and dissipation potentials

$$\begin{cases} \mathbf{W} = \frac{1}{2}E(z + \ell\sin\theta - \alpha_1)^2 + \frac{1}{2}E(z - \ell\sin\theta - \alpha_2)^2 + \frac{1}{2}h(\alpha_1)^2 + \frac{1}{2}h(\alpha_2)^2 + \lambda(z + L\cos\theta), \\ \mathbf{D} = \frac{\eta}{2}(\dot{\alpha}_1)^2 + \frac{\eta}{2}(\dot{\alpha}_2)^2. \end{cases} \tag{47}$$

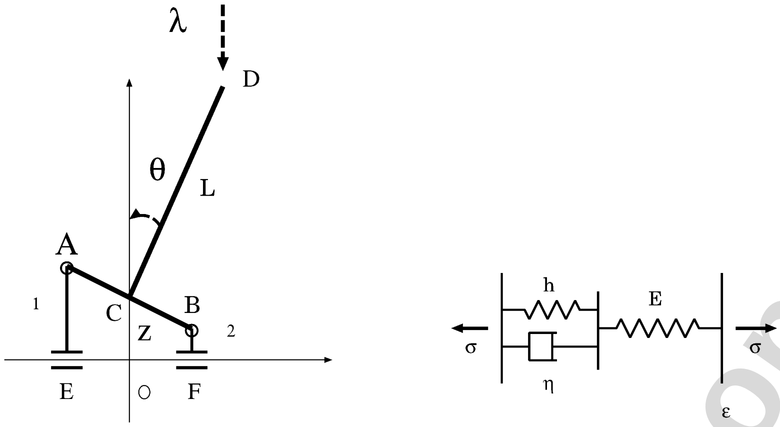

Fig. 2. The visco-elastic Shanley's column with kinematic hardening.

The quasi-static equations of the system submitted to the action of a vertical load of amplitude $\lambda(t) \in [0, \lambda_M]$ are

$$\begin{cases} E(2z - \alpha_1 - \alpha_2) + \lambda(t) = 0, \\ \ell \sin 2\theta + (\alpha_2 - \alpha_1)\cos\theta - \frac{\lambda L}{E\ell}\sin\theta = 0, \\ \eta\dot{\alpha}_1 = E(z + \ell\sin\theta - \alpha_1) - h\alpha_1, \\ \eta\dot{\alpha}_2 = E(z - \ell\sin\theta - \alpha_2) - h\alpha_2. \end{cases} \tag{48}$$

The quasi-static response of the system under the action of a varying load parameter $\lambda(t)$ is now considered. The symmetric quasi-static evolution associated with the symmetric initial condition

$$\alpha_{1o}(0) = \alpha_{2o}(0) = 0, \quad \theta_o(t) = 0, \quad z = z_o(t)$$

can be considered since the explicit expression of this solution is straightforward. Its asymptotic stability is obtained from (45) i.e. from the positivity of the matrix $\overline{\mathbf{W}}^{\mathbf{o}}_{,\mathbf{qq}}$:

$$\overline{\mathbf{W}}^{\mathbf{o}}_{,\mathbf{qq}} = \begin{bmatrix} 2E & 0 & -E & -E \\ 0 & 2E\ell^2 - \lambda L & -\ell E & \ell E \\ -E & -\ell E & h+E & 0 \\ -E & \ell E & 0 & h+E \end{bmatrix}$$

since the dissipation potential is quadratic and strictly convex. It is then concluded that the symmetric response is asymptotically stable if $\lambda(t) < \lambda_T = \frac{Eh}{E+h}\frac{2\ell^2}{L}$ for all t.

The equilibrium positions of the system under the action of a constant load λ are given by

$$\begin{cases} E(2z - \alpha_1 - \alpha_2) + \lambda = 0, \\ \ell \sin 2\theta + (\alpha_2 - \alpha_1)\cos\theta - \frac{\lambda L}{E\ell}\sin\theta = 0, \\ z + \ell\sin\theta = \frac{E+h}{E}\alpha_1, \\ z - \ell\sin\theta = \frac{E+h}{E}\alpha_2. \end{cases}$$

In function of the load parameter λ, two equilibrium curves are obtained with trivial symmetric positions, cf. Fig. 3

$$\theta = 0, \quad \alpha_1 = \alpha_2 = -\frac{\lambda}{2h}, \quad z = -\frac{\lambda}{2}\frac{E+h}{Eh}$$

and a non-symmetric positions

$$\lambda = \frac{Eh}{E+h}\frac{2\ell^2}{L}\cos\theta, \qquad z = -\frac{\lambda}{2}\frac{E+h}{Eh}.$$

The stability analysis of these equilibrium positions is straightforward from the criterion of second variation of energy. For this example, a symmetric position is asymptotically stable if $\lambda < \lambda_T$ and unstable if $\lambda > \lambda_T$ while a non-symmetric equilibrium position is unstable. Indeed, in the last case, the matrix $\overline{\mathbf{W}}^{\mathbf{o}}_{,\alpha\alpha} - \overline{\mathbf{W}}^{\mathbf{o}}_{,\alpha\mathbf{u}}\overline{\mathbf{W}}^{\mathbf{o}\,-1}_{,\mathbf{uu}}\overline{\mathbf{W}}^{\mathbf{o}}_{,\mathbf{u}\alpha}$ admits as eigenvalues $\frac{2Eh}{E+h}$ and $-L\lambda_T\sin^2\theta < 0$, thus the criterion of second variation is unsatisfied.

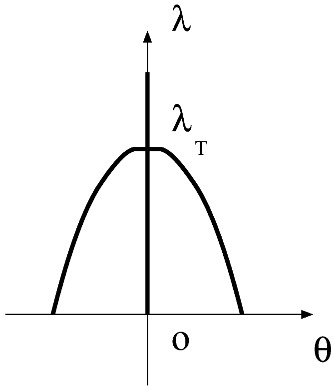

Fig. 3. Equilibrium curves of the Shanley's visco-elastic column.

5. Conclusion

In this discussion, it has been shown that the criterion of second variation of energy is a sufficient condition ensuring the stability of a quasi-static response under the perturbation of the initial conditions for a standard dissipative system.

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