Critical Plane Concept and Energy Approach in Multiaxial Fatigue

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Thierry Palin-Luc, Talence, France and Frank Morel, Angers, France

This paper aims to propose a new interpretation of several energy based critical plane high cycle fatigue criteria. More exactly, the meaning of several global energy based criteria is detailed from the calculation of normal and shear work on a critical plane. This study is limited to multiaxial fatigue under constant amplitude and synchronous loadings (proportional or non-proportional), with or without mean load. After a short review of some energy based criteria using the critical plane concept or a global approach, the reasons why energy based critical plane criteria are more and more developed in literature are summarized. It is shown that a direct link exists between critical plane energy parameters and global ones. A critical examination of the famous Findley et al. experiments is proposed and the previous link is illustrated with two mesoscopic criteria developed by Papadopoulos and the global energy based criterion of Banvillet et al. This leads to propose an interpretation of fatigue crack initiation from the mesoscopic scale up to the macroscopic scale either with stress and strain quantities or with strain energy parameters.

Many different fatigue criteria are available in the literature and can be used by the designer who deals with multiaxial fatigue. The choice of the most appropriate one is an arduous task but sometimes it can be proved that criteria built according to different approaches can lead to very similar predictions. Most of the fatigue criteria can be divided into two groups. The criteria of the first one are stress or strain based and some of them use a critical plane concept. The second one, composed of energy based criteria, can be subdivided into (i) critical plane energy based approaches and (ii) global energy based methods. A comprehensive survey of energy criteria was published in 1999 by Macha and Sonsino [1].

Global energy based approaches

A usual approach for ductile materials in high cycle fatigue (HCF) under proportional multiaxial loadings [1] is the distorsion strain energy hypothesis, known as Von Mises hypothesis. It considers that fatigue damage is the consequence of the deviatoric (distorsion) strain energy density $\Phi_{d,a} = s_{ij,a} e_{ij,a}/2$, where $s_{ij,a}$ and $e_{ij,a}$ are respectively the alternated parts of the deviatoric stress and strain tensors.

In 1974, Ellyin [2] proposed a fatigue criterion based on the cyclic shear strain energy W_f defined by: $W_f = \int_{cycle} s_{ij} de_{ij}$. This parameter is a sum of elastic and plastic shear strain energies. Several evolutions of this criterion were proposed. Lefebvre et al. [3] used the effective strain energy while Ellyin and Golos [4] considered the sum of elastic strain energy under tension and plastic energy of effective strain. Leis [5] assumed that the damaging energy parameter under proportional multiaxial fatigue with creep is the total internal strain energy $W_T = \int_{cycle} \Sigma_{ij} dE_{ij}$. This parameter includes also the plastic and the elastic strain energy density. Garud [6] assumed that plastic strain energy W_p is a relevant damage parameter to predict fatigue crack initiation. $W_p = \int_{cycle} \Sigma_{ij} dE^p_{ij}$. This energy is the sum of the normal and shear plastic strain. In 1998, Palin-Luc et al. [7] proposed, for HCF under fully reversed loadings, a volumetric energy based multiaxial fatigue criterion considering that the damage parameter is the mean value over a loading cycle of the potential elastic strain energy density: $W_a = \frac{1}{2T}\int_T \Sigma_{ij,a} E^e_{ij,a} dt$. From this proposal, Banvillet and Palin-Luc proposed in 2001 [7] - [9] an evolution of this criterion for any loadings. The damaging parameter is there the elastic strain work density (after elastic shakedown which is supposed to occur in middle cycle fatigue and high cycle fatigue [10]), W_g, given to an elementary volume by each component of the stress strain tensors: $W_g = \sum_{i,j}\int_T \langle \Sigma_{ij}(t)\dot{E}_{ij}(t)\rangle dt$.

Critical plane energy based approaches

Smith, Watson and Topper (SWT) [11] proposed to take into account the mean stress effect on the tensile fatigue strength by the product of the normal strain amplitude and the maximum normal stress. Recently, Socie [12] observed that short fatigue cracks grow on the plane perpendicular to the maximum principal stress and strain (mode I). He recommends to use the SWT damage parameter, $\sigma_{max}\varepsilon_a$, calculated on the plane where the normal strain range is maximum. An evolution of the SWT parameter was proposed by Chu et al [13] and called the specific work of shear, τ, and normal stress, σ, on the critical plane. This is defined by the sum: $2\tau_{max}\gamma_a + \sigma_{max}\varepsilon_a$. The critical plane is the material plane experiencing the maximum value of the previous sum. But, according to the authors, this paramater has not been clearly correlated with experiments. In 1993, Liu proposed to predict the multiaxial fatigue strength of materials under proportional and non-proportional loadings by using the virtual strain energy (VSE) $\Delta W_{VSE} = \Delta W_{VSE,n} + \Delta W_{VSE,\gamma} = \Delta\sigma\Delta\varepsilon + \Delta\tau\Delta\gamma/2$ computed on the critical fracture plane. This critical plane is depending on the cracking mode of the material (mode I or II) and on the crack orientation in mode II (type A or B) [14].

Glinka et al. [15] used the normal and shear parts of energy in the critical plane: $W^* = (\Delta\sigma/2)(\Delta\varepsilon/2) + (\Delta\tau/2)(\Delta\gamma/2)$. This plane experiences the maximum shear strain.

The previous review is not exhaustive, a lot of energy based parameters being available to estimate fatigue life under variable amplitude loadings. Indeed, a cycle counting algorithm is often applied to a counting variable and it is easier to extract cycles by handling a scalar than by handling a stress vector. An other reason is that the product of stress and strain related to a material plane seems appropriate to take into account the multiaxiality of stresses and strains and to predict the fatigue crack initiation or fracture plane orientation. Furthermore, by considering both stresses and strains, the plastic flow which occurs in LCF is taken into account, a combination of stress and strain is – a priori – interesting for an approach devoted to both LCF and HCF.

Global Strain Energy and Critical Plane Energy Parameters

Let us define now several energy parameters to be used in the next parts. A point O of a loaded component and a frame $(O;\underline{x},\underline{y},\underline{z})$ attached to the component are considered. The stress state at this point is defined by a stress tensor $\underline{\underline{\Sigma}}$. A plane Δ passing through O is orientated by its unit normal vector $\underline{n}$ (1) where φ, θ are the spherical co-ordinates (Figure 1).

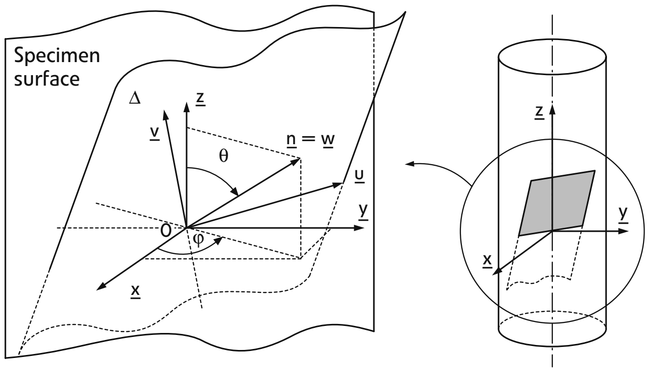

Figure 1. Coordinate systems linked to the component surface and to the material plane Δ

$$\underline{n} = (\sin\theta\cos\varphi)\,\underline{x} + (\sin\theta\sin\varphi)\,\underline{y} + (\cos\theta)\,\underline{z} \quad (1)$$

Let us now introduce a second frame $(O;\underline{u},\underline{v},\underline{w})$ where $\underline{w}$ is directed along the normal $\underline{n}$ to the plane Δ and the unit vectors $\underline{u}$, $\underline{v}$, are given by (Figure 1):

$$\underline{u} = (-\sin\varphi)\,\underline{x} + (\cos\varphi)\,\underline{y} \text{ and } \underline{v} = (-\cos\theta\cos\varphi)\,\underline{x} + (-\cos\theta\sin\varphi)\,\underline{y} + (\sin\theta)\,\underline{z} \quad (2)$$

On the plane Δ the stress vector $\underline{\Sigma}_{\underline{n}}(t)$, depending on time t under cyclic loading, is given by:

$$\underline{\Sigma}_{\underline{n}}(t) = \underline{\underline{\Sigma}}(t)\cdot\underline{n} \quad (3)$$

$\underline{\Sigma}_{\underline{n}}(t)$ is composed of a normal stress vector $\underline{\sigma}(t)$, and a shear stress vector $\underline{\tau}(t)$:

$$\underline{\Sigma}_{\underline{n}}(t) = \underline{\sigma}(t) + \underline{T}(t) \quad (4)$$

$$\text{where } \underline{\sigma}(t) = (\underline{\Sigma}_{\underline{n}}(t)\cdot\underline{n})\underline{n} \quad \text{and} \quad \underline{\tau}(t) = (\underline{\Sigma}_{\underline{n}}(t) - \underline{\sigma}(t) \quad (5)$$

Now, if the strain state at the same point O is defined by the tensor $\underline{\underline{E}}(t)$, the strain vector $\underline{E}_{\underline{n}}(t)$ on the material plane Δ can be defined by:

$$\underline{E}_{\underline{n}}(t) = \underline{\underline{E}}(t)\cdot\underline{n} \quad (6)$$

This vector is composed of a longitudinal strain vector $\underline{\varepsilon}(t)$ and a shear part $\underline{\gamma}(t)$ as well:

$$(\underline{E}_{\underline{n}}(t) - \underline{\varepsilon}(t) + \underline{Y}(t) \quad (7)$$

$$\text{where } \underline{\varepsilon}(t) = (\underline{E}_{\underline{n}}(t)\cdot\underline{n})\underline{n} \quad \text{and} \quad \underline{Y}(t) = (\underline{E}_{\underline{n}}(t) - \underline{\varepsilon}(t) \quad (8)$$

From the aforementioned quantities (eq. 4) and (eq. 7), at any time the strain energy density parameter $\psi_{\underline{n}}(t)$ related to the material plane Δ can be defined by:

$$\psi_{\underline{n}}(t) = \underline{\Sigma}_{\underline{n}}(t)\cdot(\underline{E}_{\underline{n}}(t) \Rightarrow \psi_{\underline{n}}(t) = \underline{\sigma}(t)\cdot\underline{\varepsilon}(t) + \underline{T}(t)\cdot\underline{Y}(t) \quad (9)$$

If a normal part $\psi_{\underline{n},\varepsilon}$ and a shear part $\psi_{\underline{n},\gamma}$ of the strain energy density parameter $?_{\underline{n}}$ for the material plane Δ are introduced, it comes:

$$\psi_{\underline{n}}(t) = \psi_{\underline{n},\varepsilon}(t) + \psi_{\underline{n},Y}(t) \quad (10)$$

$$\text{where } \psi_{\underline{n},\varepsilon}(t) = \underline{\sigma}(t)\cdot\underline{\varepsilon}(t) \quad \text{and} \quad \psi_{\underline{n},Y}(t) = \underline{T}(t)\cdot\underline{Y}(t) \quad (11)$$

For an homogeneous material with an isotropic elastic behaviour at the macroscopic scale, let us now try to establish the relationships between these energy quantities related to a material plane and the potential elastic strain energy density $\Phi(t) = \Sigma_{ij}(t)\,E_{ij}(t)/2$ which can be expressed as the sum of two parts: the distortion $\Phi_d(t)$ and spherical $\Phi_s(t)$ parts defined as:

$$\Phi_d(t) = \frac{1+\nu}{E} J_2(t) \quad \text{and}$$

$$\Phi_s(t) = \frac{3}{2}\left(\frac{1-2\nu}{E}\right) \Sigma_H^2(t) \tag{12}$$

where J_2 is the second invariant of the deviatoric stress tensor, Σ_H is the hydrostatic stress, E is the Young modulus, and ν is the Poisson ratio of the material.

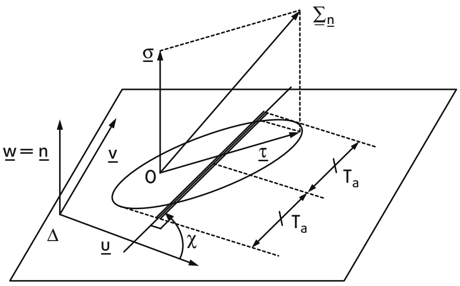

Figure 2. Resolved shear stress amplitude on the material plane Δ for a sinusoidal loading

To build this link, it is necessary to carry out an integration over all the possible material planes passing through the point O. In the following sections the notation $\hat{X}$ will correspond to this integration. $\hat{X}$ is nothing but the volumetric mean value of X(t).

$$\hat{X}(t) = \frac{1}{4\pi} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} X(t) \sin\theta \, d\theta \, d\varphi \tag{13}$$

Let us then apply this operation to the elastic strain energy density parameter $?_{\underline{n}}(t)$:

$$\hat{\psi}_{\underline{n}}(t) = \hat{\psi}_{\underline{n}\varepsilon}(t) + \hat{\psi}_{\underline{n}\Upsilon}(t) \tag{14}$$

$$\text{where } \hat{\psi}_{\underline{n},\varepsilon}(t) = \frac{1}{4\pi} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \underline{\sigma}(t) \cdot \underline{\varepsilon}(t) \sin\theta \, d\theta \, d\varphi$$

$$\text{and } \hat{\psi}_{\underline{n},\Upsilon}(t) = \frac{1}{4\pi} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \underline{T}(t) \cdot \underline{T}(t) \sin\theta \, d\theta \, d\varphi \tag{15}$$

After some long calculations, it can be demonstrated in Annex 1 that:

$$\hat{\psi}_{\underline{n},\Upsilon}(t) = \frac{2}{5} \Phi_d(t) \quad \text{and}$$

$$\hat{\psi}_{\underline{n},\varepsilon}(t) = \frac{2}{3}\left[\Phi_s(t) + \frac{2}{5}\Phi_d(t)\right] \tag{16}$$

and hence

$$\hat{\psi}_{\underline{n}}(t) = \frac{2}{3}\left[\Phi_s(t) + \Phi_d(t)\right] = \frac{2}{3}\Phi(t) \tag{17}$$

Eqs. 16 and 17 prove that a direct link exists at any time between the different energy parameters on a material plane and the corresponding "global" potential strain energy densities Φ_s and Φ_d usually defined. However, while the volumetric average of the shear strain energy parameter related to a plane $\hat{\psi}_{\underline{n}\Upsilon}$ is simply proportional to the distortion strain energy density Φ_d, the volumetric average of the normal strain energy density $\hat{\psi}_{\underline{n},\varepsilon}$ depends both on the distortion and spherical parts. This tends to indicate that the energy related to a plane must be handled very carefully especially when dealing with its normal part.

Alternating part and mean value of a cyclic loading do not play the same role in fatigue crack initiation. That is the reason why the alternating part and mean values of the previous energy quantities have to be separated to demonstrate the link existing between stress based critical plane fatigue criteria and energy based ones. Let us assume that the stress tensor is periodic with a mean value $\Sigma_{ij,m}$ and an alternating part $\Sigma_{ij,a}(t)$; its time evolution may be sinusoidal, triangular or square (eq. 18), where "triang" and "sqr" are respectively the fully reversed triangular and square periodic functions versus time.

$$\begin{aligned} \Sigma_{ij}(t) &= \Sigma_{ij,m} + \Sigma_{ij,a}\sin(\varpi t - \varphi_{ij}) \text{ or} \\ \Sigma_{ij}(t) &= \Sigma_{ij,m} + \Sigma_{ij,a}\operatorname{triang}(\varpi t - \varphi_{ij}) \text{ or} \\ \Sigma_{ij}(t) &= \Sigma_{ij,m} + \Sigma_{ij,a}\operatorname{sqr}(\varpi t - \varphi_{ij}) \end{aligned} \tag{18}$$

From the previous equations, the alternating and mean value of the normal and shear stress or strain vectors can be defined.

$$\begin{aligned} \underline{\Sigma}_{\underline{n}}(t) &= \underline{\sigma}_m + \underline{\sigma}_a(t) + \underline{T}_m + \underline{T}_a(t) \text{ and} \\ \underline{E}_{\underline{n}}(t) &= \underline{\varepsilon}_m + \underline{\varepsilon}_a(t) + \underline{\Upsilon}_m + \underline{\Upsilon}_a(t) \end{aligned} \tag{19}$$

It is then possible to define two parts of the normal strain energy density (respectively the shear strain energy density). One is due to mean stress and strain, the other corresponds to alternating stress and strain.

$$\psi_{\underline{n},\varepsilon,m} = \underline{\sigma}_m \cdot \underline{\varepsilon}_m \text{ and } \psi_{\underline{n},\varepsilon,a}(t) = \underline{\sigma}_a(t) \cdot \underline{\varepsilon}_a(t) \tag{20}$$

$$\psi_{\underline{n},\gamma,m} = \underline{\tau}_m \cdot \underline{\gamma}_m \text{ and } \psi_{\underline{n},\gamma,a}(t) = \underline{\tau}_a(t) \cdot \underline{\gamma}_a(t) \tag{21}$$

But it has to be pointed out that $\psi_{\underline{n},\varepsilon}(t) \neq \psi_{\underline{n},\varepsilon,m} + \psi_{\underline{n},\varepsilon,a}(t)$ and $\psi_{\underline{n},\gamma}(t) \neq \psi_{\underline{n},\gamma,m} + \psi_{\underline{n},\gamma,a}(t)$ because of the coupling terms: $\underline{\sigma}_m \cdot \underline{\varepsilon}_a(t)$, $\underline{\sigma}_a(t) \cdot \underline{\varepsilon}_m$ and $\underline{T}_m \cdot \underline{\Upsilon}_a(t)$, $\underline{T}_a(t) \cdot \underline{\Upsilon}_m$ occurring in the scalar product $\underline{\Sigma}_{\underline{n}}(t) \cdot \underline{E}_{\underline{n}}(t)$. Let us recall that because of the orthogonality property of the vectors $\underline{\sigma}(t)$ (respectively $\underline{\varepsilon}(t)$) and $\underline{\gamma}(t)$ (respectively $\underline{\tau}(t)$) the following equalities are verified:

$$\begin{aligned} &\underline{\sigma}_m \cdot \underline{\Upsilon}_a(t) + \underline{T}_m \cdot \underline{\varepsilon}_a(t) = \underline{\sigma}_a(t) \cdot \underline{\Upsilon}_m = \\ &= \underline{T}_a(t) \cdot \underline{\varepsilon}_m = \underline{\sigma}_m \cdot \underline{\Upsilon}_m = \underline{\sigma}_a(t) \cdot \underline{\Upsilon}_a(t) = \\ &= \underline{T}_m \cdot \underline{\sigma}_m \cdot \underline{T}_a(t) \cdot \underline{\varepsilon}_a(t) = 0 \end{aligned}$$

$$\begin{aligned} &\underline{\Sigma}_{\underline{n}}(t) \cdot (\underline{E}_{\underline{n}}(t) = [\underline{\sigma}_m + \underline{\sigma}_a(t) + \underline{T}_m + \underline{T}_a(t)] \cdot \\ &\cdot [\underline{\varepsilon}_m + \underline{\varepsilon}_a(t) + \underline{\Upsilon}_m + \underline{\Upsilon}_a(t)] = \\ &= [\underline{\sigma}_a(t) \cdot \underline{\varepsilon}_a(t) + \underline{T}_a(t) \cdot \underline{\Upsilon}_a(t)] + \\ &+ [\underline{\sigma}_m \cdot \underline{\varepsilon}_m + \underline{T}_m + \underline{\Upsilon}_m] + \\ &+ [\underline{\sigma}_m \cdot \underline{\varepsilon}_a(t) + \underline{T}_m \cdot \underline{\Upsilon}_a(t) + \underline{\sigma}_a(t) \cdot \underline{\varepsilon}_m + \underline{T}_a \cdot \underline{\Upsilon}_m(t)] \end{aligned} \tag{22}$$

From this analysis, it is clear now that the energy parameter proposed by Glinka et al. [15] is nothing but the sum of $\psi_{\underline{n},\varepsilon,a}$ and $\psi_{\underline{n},\gamma,a}$ on the plane of maximum shear strain.

Critical Plane Energy Approach

As demonstrated in 1993 by Papadopoulos [16], for any synchronous multiaxial sinusoidal loading: $\Sigma_{ij}(t) = \Sigma_{ij,m} + \Sigma_{ij,a} \sin(\omega t - \phi_{ij})$, the shear stress vector $\underline{\tau}(t)$ acting on the material plane Δ orientated by $\underline{n}$, can be written as a combination of $\tau_u(t)$ and $\tau_v(t)$ in the frame $(O;\underline{u},\underline{v},\underline{w})$ (Figure 2) (eq. 23) where A, B, C, and D are function of the phase differences ϕ_{ij} and of the stress amplitudes $\Sigma_{ij,a}$.

$$\begin{aligned} \underline{T}(t) &= T_u(t)\,\underline{u} + T_v(t)\,\underline{v} \text{ with } T_u(t) = \\ &= A \sin\omega t + B \cos\omega t + T_{u,m} \text{ and } T_v(t) = \\ &= C \sin\omega t + D \cos\omega t + T_{v,m} \end{aligned} \tag{23}$$

Critical Plane Strain Energy Density

Let us consider now the alternating part of the shear strain energy density parameter $\psi_{\underline{n},\gamma,a}(t) = \underline{\tau}_a(t) \cdot \underline{\gamma}_a(t)$. For an isotropic material with a linear elastic behaviour at the macroscopic scale it is easy to prove that $\underline{\Upsilon}_a(t) = \frac{1+\nu}{E}\underline{T}_a(t)$. By using this remark its can be shown (Annex 2) that the mean value on a loading period of $\psi_{\underline{n},\gamma,a}(t)$ is simply depending on the previous functions A, B, C, D (eq. 24).

$$\begin{aligned} &\frac{1}{T}\int_0^T \psi_{\underline{n},\gamma,a}(t)\,dt = \frac{1+\nu}{E}\frac{1}{T}\int_0^T \underline{\tau}_a(t) \cdot \underline{\tau}_a(t)\,dt = \\ &= \frac{1+\nu}{E}\left(\frac{A^2+B^2+C^2+D^2}{2}\right) \end{aligned} \tag{24}$$

It is very interesting to note that the term in the right hand side of eq. 24 is proportional to $(T_\sigma)^2$, where T_σ is a parameter proposed by Papadopoulos in

1993 [17]. T_σ is the mean value of the macroscopic resolved shear stress amplitude T_a acting on all the possible directions (located by the angle χ) of a material plane Δ (Figure 2).

$$T_\sigma(\theta,\varphi)=\sqrt{\int_{\psi=0}^{2\pi} T_a^2(\theta,\varphi,\chi)\,d\chi} \qquad (25)$$

Indeed, Papadopoulos showed that T_σ is equal to:

$$T_\sigma(\theta,\varphi)=\sqrt{\pi}\sqrt{A^2+B^2+C^2+D^2} \qquad (26)$$

Thus, for any synchronous multiaxial sinusoidal loading, eq. 27 relates T_σ and the alternating shear strain energy density parameter on the material plane Δ.

$$T_\sigma(\theta,\varphi)=\sqrt{\frac{2\pi E}{1+\nu}}\sqrt{\frac{1}{T}\int_T \psi_{\underline{n},\Upsilon,a}(t)\,dt} \qquad (27)$$

For "ductile" metals (i.e. if $0.5 \le \tau^D_{-1}/\sigma^D_{ten,-1} \le 0.6$) Papadopoulos proposed an endurance criterion using T_σ and the maximum hydrostatic stress on a loading period. Since the effect of this last parameter is very important in fatigue crack initiation, the criterion is a linear combination (eq. 28) where a and b are two material parameters identified with two experimental fatigue limits.

$$\max_{\theta,\varphi}\,[T_\sigma(\theta,\varphi)] + a\Sigma_{H,max} \le b \qquad (28)$$

From eq. 16 it is now clear that the "Papadopoulos T_σ criterion" (eq. 28) can be written as follows (eq. 29) with energy parameters, where $I_1(t = \Sigma_{ii}(t)$:

$$\max_{\theta,\varphi}\left[T_\sigma(\theta,\varphi)\right]+a\Sigma_{H,max} \le b \;\Leftrightarrow$$
$$\Leftrightarrow\; \max_{\theta,\varphi}\left\{\sqrt{\frac{2\pi E}{1+\nu}}\sqrt{\frac{1}{T}\int_T \psi_{\underline{n},\Upsilon,a}(t)\,dt}\right\}+ \qquad (29)$$
$$+a\,\max_{t\in T}\left[\frac{I_1(t)}{|I_1(t)|}\sqrt{\frac{2E}{3(1-2\nu)}\Phi_s(t)}\right]\le b$$

For "hard" metals (i.e. if $0.6 \le \tau^D_{-1}/\sigma^D_{ten,-1} \le 0.8$), Papadopoulos proposed an other criterion (eq. 30).

$$M_\sigma + c\Sigma_{H,max} \le d \qquad (30)$$

Where the M_σ quantity is nothing else but the volumetric mean value of T_σ over all the material planes of an elementary volume (eq. 31). This "global" quantity was defined by Papadopoulos because fatigue crack initiation is assumed to be strongly dependent on the shear stress amplitude, not only over all the possible sliding directions on one material plane of a grain (or a cristal), but also over all the planes of a grain.

$$M_\sigma=\sqrt{\frac{5}{8\pi^2}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi} T_\sigma^2(\theta,\varphi)\sin\theta\,d\theta\,d\varphi} \qquad (31)$$

In 2001, Morel et al. [18] proved that for any sychronous sinusoidal multiaxial loadings the second Papadopoulos criterion can be easily written with energy quantities (eq. 32).

$$M_\sigma + c\Sigma_{H,max} \le d \;\Leftrightarrow\; \pi\sqrt{\frac{2E}{(1+\nu)}W_{a,d}}\,+ \qquad (32)$$
$$+c\,\max_{t\in T}\left[\frac{I_1(t)}{|I_1(t)|}\sqrt{\frac{2E}{3(1-2\nu)}\Phi_s(t)}\right]\le d$$

Where $W_{a,d}$ is the deviatoric part of the mean value over a loading cycle of the potential strain energy density due to alternated stresses (eq. 33) as used in the volumetric energy based high cycle multiaxial fatigue criterion proposed by Palin-Luc and Lasserre [7].

$$W_{a,d}=\frac{1}{T}\int_T \Phi_{d,a}(t)\,dt=\left(\frac{1+\nu}{E}\right)\frac{1}{T}\int_T J_{2,a}(t)\,dt \qquad (33)$$

the material $Wg_{\underline{n},\gamma}$ (i.e. the integral over the load cycle of the positive part of the shear strain power).

$$Wg_{\underline{n},\gamma}=\int_T\left\langle \dot{\psi}_{\underline{n},\gamma}(t)\right\rangle dt \qquad (34)$$

Around the fatigue limit, the mean shear stress is usualy considered as non damaging if the maximum shear stress stays below the yield shear stress. For reversed loading, $Wg_{\underline{n},\gamma}$ is given by the following equation (Annex 3):

$$Wg_{\underline{n},\gamma,a}=\int_T\left\langle 2\underline{\tau}_a(t)\cdot\dot{\underline{\gamma}}_a(t)\right\rangle dt = \frac{4(1+\nu)}{E}\sqrt{G^2+H^2} \qquad (35)$$

where $G = (A^2 - B^2 + C^2 - D^2)/2$ and $H = (AB + CD)$, A, B, C, D being the functions proposed by Papadopoulos; they depend on the stress amplitudes and the phase shifts [16].

For any proportional sinusoidal loadings, it is easy to demonstrate from eq. 26 and eq. 35 that T_σ is proportional to $\sqrt{Wg_{\underline{n},\gamma,a}}$.

$$T_\sigma=\sqrt{\pi}\sqrt{\frac{E}{2(1+\nu)}Wg_{\underline{n},\gamma,a}} \qquad (36)$$

This means that the material planes experiencing the maximum value of T_σ and $\sqrt{Wg_{\underline{n},\gamma,a}}$ are the same. As illustrated in Figure 3, in many loading cases these planes are not unique; for instance in tension along $\underline{z}$, these critical planes are defined by: $(\forall\varphi,\ \theta = \pi/4)$, $(\forall\varphi,\ \theta = 3\pi/4)$ and for a pure shear in the plane $\underline{y}$, $\underline{z}$: $(\forall\varphi,\ \theta = 0)$, $(\forall\varphi,\ \theta = \pi)$, $(\varphi = \theta = \pi/2)$ and $(\varphi = 3\pi/2,\ \theta = \pi/2)$. For non proportional loadings, a simple relation between T_σ and $\sqrt{Wg_{\underline{n},\gamma,a}}$ has not yet been found but further works will investigate this particular loading case.

Figure 3: T_σ normalized distribution versus (φ, θ) in tension (a) along $\underline{z}$ and in torsion (b) (pure shear in the plane $(\underline{y}, \underline{z})$).

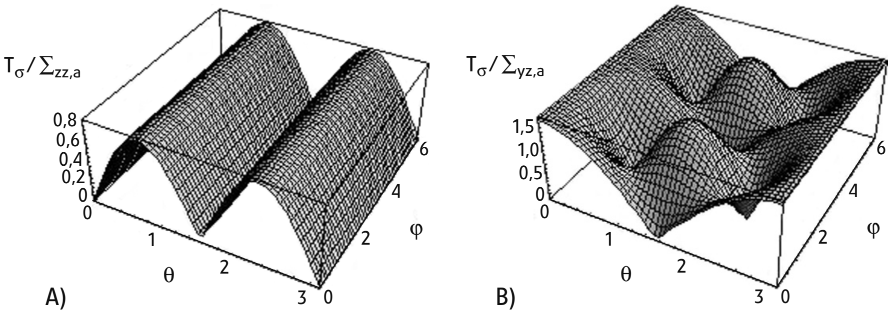

Figure 3. T_σ normalized distribution versus (φ, θ) in tension (a) along $\underline{z}$ and in torsion, (b) (pure shear in the plane ($\underline{y}$, $\underline{z}$))

Strain work given parameters on a critical plane

Using the idea of Banvillet et al. [9],[10], it is assumed that fatigue crack initiation is due to the strain work density given to (ii) either only one material plane Δ of the considered elementary volume of material, or (ii) to all the material planes of this volume.

Thus, let us consider on the plane Δ the shear strain work density given to

Since the critical planes according to T_σ or $\sqrt{Wg_{\underline{n},\gamma,a}}$ are not always unique, it is interesting to consider all the planes of an elementary volume of material. In this case, the shear strain work Wg_γ and the normal strain work Wg_ε given to all the planes within an elementary volume are respectively defined by equations (eq. 37).

$$Wg_\gamma = \int_T \left\langle \dot{\psi}_{\underline{n},\gamma}(t) \right\rangle dt \text{ and}$$
$$Wg_\varepsilon = \int_T \left\langle \dot{\psi}_{\underline{n},\varepsilon}(t) \right\rangle dt \tag{37}$$

From eq. 16 it is easy to prove that:

$$Wg_\gamma = \int_T \left\langle \frac{2}{5}\dot{\Phi}_d(t) \right\rangle dt \text{ and}$$
$$Wg_\varepsilon = \int_T \left\langle \frac{2}{3}\left(\dot{\Phi}_s(t) + \frac{2}{5}\dot{\Phi}_d(t) \right) \right\rangle dt \tag{38}$$

Eq. 38 means that Wg_γ is also the sum over the load cycle of the positive variations of the distortion part of the potential strain energy density $\Phi(t)$, and Wg_ε is a combination of the positive variations of both the spherical and the deviatoric parts of $\Phi(t)$.

Discussion

Findley's experiments. In a well known paper, Findley et al. [19] describe a fatigue test on a rotating disk loaded in diametral compression by rollers. Fatigue cracks initiated at the center of this disk. In this area there was a biaxial stress state and principal direction of stresses were rotating in a frame linked with the specimen. Because, at this location, the potential strain energy density $\Sigma_{ij}(t)E_{ij}(t)/2$ was constant over one rotation of the disk, the authors concluded that the energy can not be used to predict fatigue crack initiation. For instance, with a static tension test it is possible to submit a specimen to the same value of the potential strain energy density as during the Findley et al. fatigue test, without crack initiation (the static load required to initiate a crack is higher). Thus for many years, this paper was used by several authors to claim that energy based approaches are not appropriate to predict the fatigue strength of materials and structures. But this was a premature conclusion.

Of course, the potential strain energy density is constant in the previous experiment and the energy parameter proposed by Leis [5] is not relevant to fatigue. However, Park and Nelson [20] demonstrated that the distorsion work is not null nor constant for the Findley et al. test. They refute the negative conclusion of Findley et al. concerning energy based approaches for fatigue. On a similar way, Banvillet et al. [10] showed that the strain work density given to the material per loading period is adapted for fatigue prediction. If a critical plane energy based parameter is now considered, the following results are obtained. At the center of the disk of the Findley's tests, the stress state can be written as [20]:

$\sigma_{11}(t) = 2.9\,\sigma \sin(\omega t) - 1.1\sigma$,
$\sigma_{11}(t) = 2.9\,\sigma \sin(\omega t - \pi) - 1.1\sigma$,
$\sigma_{12}(t) = 2.9\,\sigma \sin(\omega t - \pi/2)$

where σ is a constant. For such a stress state one can demonstrate that the spherical and deviatoric parts of the potential strain energy density are not time dependent (eq. 39) as shown by Findley et al. for the total potential strain energy density $\Phi(t)$ [3, 7].

$$\Phi_s(t) = 2\left(\frac{1-2\nu}{3E}\right)(1.1\sigma)^2 \text{ and}$$
$$\Phi_d(t) = \left(\frac{1+\nu}{3E}\right)\left[3(2.9\sigma)^2 + (1.1\sigma)^2\right] \tag{39}$$

In the same way, the normal and shear parts ($\psi_{\underline{n},\varepsilon}$ and $\psi_{\underline{n},\gamma}$) of the elastic strain energy density parameter $\psi_{\underline{n}}$ related to a material plane Δ orientated by (θ,φ) are fluctuating with time (eqs. 40 and 41) during the experiment of Findley. $\psi_{\underline{n},\varepsilon}$ and $\psi_{\underline{n},\gamma}$ are cyclic during one rotation of the disk (Figure 4, on a particular plane). Thus, these quantities can be used as damage parameters in fatigue. Banvillet et al. obtained the same conclusion with the strain work density due to each stress strain components [10].

$$\psi_{\underline{n},\varepsilon}(t) = \frac{-\sigma^2\sin^2\theta}{2E}\left[-1.1 - 2.9\sin(2\varphi - \omega t)\right]$$
$$\left[-1.1\{-1+3\nu+(1+\nu)\cos(2\theta)\} + \right. \tag{40}$$
$$\left. +5.8(1+\nu)\sin^2\theta\sin(2\varphi-\omega t)\right]$$

$$\psi_{\underline{n},\gamma}(t) = \frac{(1+\nu)\sigma^2}{4E}\left\{\left[27.65+10.83\cos(2\theta)\right]\right.$$
$$\sin^2\theta + 16.82\cos(4\varphi - 2\omega t)\sin^4\theta + \tag{41}$$
$$\left. +6.38\sin^2(2\theta)\sin(2\varphi-\omega t)\right\}$$

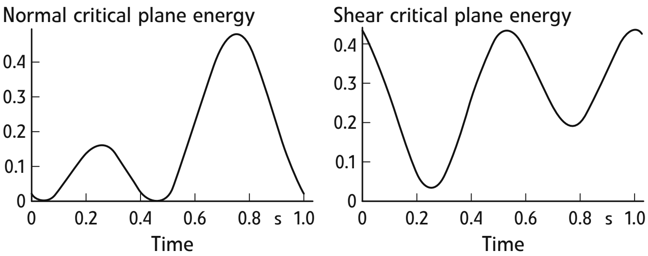

Figure 4. Evolution during a loading period of the normal, $\Psi_{\underline{n}\varepsilon}$, and shear, $\Psi_{\underline{n}y}$, critical plane energy densities on the plane ($\theta = \pi/3$ $\varphi = \pi$) for the Findley test

However, it can be proved for this test that eq. 38 and eq. 39 lead to $Wg_\gamma = Wg_\varepsilon = 0$. This means that Wg_γ and Wg_ε are not able to reflect the fatigue damage accumulation for that kind of test.

Application to combined tension and torsion loading. Even if $Wg_{\underline{n},\gamma,a}$ and T_σ are not simply linked under non-proportional loadings, the material planes experiencing the maximum values of these two quantities can be the same (Figure 5). For a combined tension and torsion sinusoidal fatigue test with a phase shift of 90°, this Figure shows that the critical planes corresponding to $Wg_{\underline{n},\gamma,a}$ and T_σ are the same if $\Sigma_{yz,a}/\Sigma_{zz,a} = 2$ (Figure 5a), but they are different if $\Sigma_{yz,a}/\Sigma_{zz,a} = 1/2$ (Figure 5b). Nevertheless, the difference between the maxima of T_σ in Figure 5b) and its values for the $Wg_{\underline{n},\gamma,a}$ critical planes is not important: a lot of planes have a high T_σ value (between 100 and 120 MPa), many of them are in coincidence with the $Wg_{\underline{n},\gamma,a}$ critical planes. Experiments have to be done to try to observe the fatigue crack initiation plane (not the fracture plane) in such test conditions.

Energy based interpretation of fatigue crack initiation at the mesoscopic scale. In his mesoscopic description of fatigue crack initiation in a polycrystaline material, Papadopoulos [17] assumed that the plasticity criterion of each grain is the Schmid's law and that each crystal follows a combined isotropic and kinematical hardening (modulus g) behaviour. Under the assumption that, at the endurance limit, the plastically deformed crystal reaches an elastic shake-down state, this author found, for "ductile metals" (i.e. if

$0.5 \leq \tau^{D}_{-1}/\sigma^{D}_{ten,-1} \leq 0.6$), an upper bound estimation, T_σ/g, of the plastic strain accumulated by the plastically less resistant grains of the elementary volume (grains with their easier glide planes parallel to the critical plane Δ). Eq. 27 proves that, for synchronous multiaxial sinusoidal loadings, this threshold value of the mesoscopic plastic strain accumulated is also proportionnal to the square root of the mean value, over a loading period, of the alternating part of the shear strain energy parameter on the critical plane $\psi_{\underline{n},\gamma,a}$. This equation shows also, that the integration carried out over a loading cycle of an energy based parameter can lead to results similar to an integration of the resolved shear stress amplitude over each possible slip direction of the critical plane. The link between integration in time of an energy based quantity and integration in space on a material plane has already been noticed by Morel et al. [18]. This is illustrated by eq. 31 and eq. 32 which prove that the integration, over a loading cycle, of the distorsion potential strain energy density due to the alternated stresses gives a result proportional to the integration of the resolved shear stress amplitude over each possible slip direction of every material planes in a point.

For "hard metals" (i. e. if $0.6 \leq \tau^{D}_{-1}/\sigma^{D}_{ten,-1} \leq 0.8$) Papadopoulos proved that M_σ/g is an upper bound limitation of the mesoscopic plastic strain accumulated by all the gliding crystals within an elementary volume. Eq. 32 shows that for any synchronous multiaxial loadings, $\sqrt{W_{a,d}}$ can be also considered as an upper bound estimation of the previous quantity since it is proportional to M_σ/g. Banvillet shows [8] that the strain work density given to an elementary volume of material $W_g = \sum_{i,j} \int_T \langle \Sigma_{ij}(t)\dot{E}_{ij}(t)\rangle \, dt$, is proportional to the mean value over a loading period of the potential elastic strain energy density W_a ($Wg = 4W_a$) for any loadings, except for out-of-phase biaxial tension (for phase shift different from 0 and π) [8]. It is shown in [18] that W_a is the sum of a spherical and a deviatoric parts: $W_a = W_{a,d} + W_{a,s}$). For Wg, the same type of relation can be demonstrated ($Wg = Wg_d + Wg_s$) except for out-of-phase biaxial tension. This leads to: $Wg_d = 4W_{a,d}$, except for out-of-phase biaxial tension. As a consequence, $\sqrt{Wg_d}$ can be also considered as an upper bound limit of the mesoscopic plastic strain accumulated by all the gliding crystals within an elementary volume under cyclic loading, except in out-of-phase biaxial tension. In future works, these laboratory test conditions have to be studied carrefully to propose a new definition of the strain work energy density given to the material under out-of-phase biaxial tension.

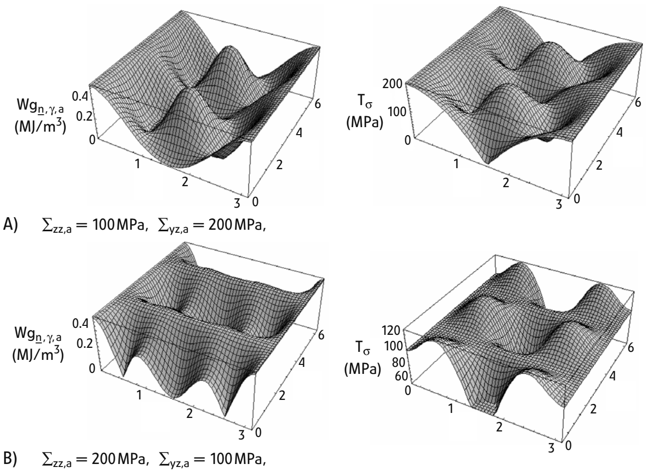

Figure 5. $Wg_{\underline{n},\gamma,a}$ and T_σ distributions on the different planes orientated by $\underline{n}$ (θ, φ), for two combined tension and torsion sinusoidal out-of-phase ($\varphi = \pi/2$) fatigue tests

Conclusion and prospects

This paper shows that, in a lot of multiaxial fatigue test conditions, several critical plane stress quantities can be related to critical plane energy based parameters or global energy based ones. Moreover, several new energy parameters related to a material plane have been introduced. By a proper integration over all the material planes, these quantities can be linked to the well-known deviatoric and spherical strain energy densities. The main important result of this work is that the mesoscopic critical plane endurance criteria proposed by Papadopoulos can be readily interpreted in terms of energy quantity. Of course, many experimental investigations under multiaxial loading have still to be done especially under out-of-phase biaxial tension. Such tests would allow to discrimate stress (or strain) and energy (or work) approaches for instance through the influence of the phase shift on the fatigue life. But this work is very promising especially if the low cycle fatigue regime is planned to be investigated. Indeed, in this fatigue regime, the use of an energy approach seems the more appropriate because both the stress and the strain quantities can play an important role in the damage mechanisms.

Nomenclature

$(O;\underline{x},\underline{y},\underline{z})$	Orthogonal cartesian coordinate system linked to the component
$(O;\underline{u},\underline{v},\underline{w})$	Orthogonal cartesian coordinate system linked to the material plane
Δ	orientated by the unit normal vector $\underline{n}$
$\underline{\underline{\Sigma}}$	stress tensor
$\underline{\underline{E}}$	strain tensor
Σ_{ij}	(i,j) component of the stress tensor $\underline{\underline{\Sigma}}$. (i, j = x, y, z)
Σ_H	hydrostatic stress (i. e. $\Sigma_H = \Sigma_{kk}/3$)
E_{ij}	(ij) component of the srain tensor $\underline{\underline{E}}$. (i, j = x, y, z)
E	Young modulus
$\underline{n}$	unit normal vector orientating the material plane Δ
φ, θ	spherical co-ordinates of unit normal vector $\underline{n}$
$\underline{\Sigma}_{\underline{n}}$	stress vector acting on the material plane Δ
$\underline{E}_{\underline{n}}$	strain vector acting on the material plane Δ
ν	Poisson ratio
$\underline{\sigma}$	normal stress vector acting on the material plane Δ
$\underline{\tau}$	shear stress vector acting on the material plane Δ
$\underline{\varepsilon}$	normal strain vector acting on the material plane Δ
$\underline{\gamma}$	shear strain vector acting on the material plane Δ
Φ	volumetric density of the potential elastic strain energy
Φ_d	distortion part of Φ
Φ_s	spherical part of Φ
$\psi_{\underline{n}}$	elastic strain energy density parameter defined on the material plane Δ orientated by $\underline{n}$
$\psi_{\underline{n},\varepsilon}$	normal part of $\psi_{\underline{n}}$
$\psi_{\underline{n},\gamma}$	shear part of $\psi_{\underline{n}}$
J_2	second invariant of the stress deviatoric tensor
I_1	first invariant of the stress tensor, $I_1 = \Sigma_{kk}$
$\sigma^{D}_{ten,-1}$	fully reversed endurance limit in tension
τ^{D}_{-1}	fully reversed endurance limit in torsion
T_σ	mean value of the macroscopic resolved shear stress amplitude on the material plane
M_σ	volumetric mean value of T_σ over all the material planes
a, b	material parameters of the Papadopoulos T_σ criterion
c, d	material parameters of the Papadopoulos M_σ criterion

APPENDICES

Annex 1: Calculation of $\hat{\psi}_{\underline{n},\varepsilon}$ and $\hat{\psi}_{\underline{n},Y}$. For any multiaxial loading of an isotropic material with a linear elastic behaviour, the distorsion part $\Phi_d(t)$ and the spherical part $\Phi_s(t)$ of the potential elastic strain energy density $\Phi(t) = \Sigma_{ij}(t)E_{ij}(t)/2$ are:

$$\Phi_d(t)=\frac{1+\nu}{E}J_2(t)=\frac{1+\nu}{3E}\left[\Sigma_{xx}^2+\Sigma_{yy}^2+\Sigma_{zz}^2+3\left(\Sigma_{xy}^2+\Sigma_{yz}^2+\Sigma_{xz}^2\right)-\left(\Sigma_{xx}\Sigma_{yy}+\Sigma_{xx}\Sigma_{zz}+\Sigma_{yy}\Sigma_{zz}\right)\right]$$

$$\Phi_s(t)=\frac{3}{2}\left(\frac{1-2\nu}{E}\right)\Sigma_H^2(t)=\frac{1-2\nu}{6E}\left(\Sigma_{xx}+\Sigma_{yy}+\Sigma_{zz}\right)^2.$$

On the material plane Δ orientated by $\underline{n}$, the shear stress vector $\underline{\tau}(t)$ and the shear strain vector $\underline{\gamma}(t)$ are respectively: $\underline{T}(t) = \underline{\Sigma}_{\underline{n}}(t) - \underline{\sigma}(t)$ and $\underline{\gamma}(t) = \underline{E}_{\underline{n}}(t) - \underline{\varepsilon}(t)$. Their components have the following expressions in the frame $R = (O; \underline{x}, \underline{y}, \underline{z})$.
Note: to simplify the following equations, (t) is omitted for each $\Sigma_{ij}(t)$ and $E_{ij}(t)$ term.

$$\underline{\sigma}(t)=\begin{pmatrix}\Sigma_{xz}\cos\theta+\Sigma_{xx}\cos\varphi\sin\theta+\Sigma_{xy}\sin\varphi\sin\theta\\ \Sigma_{yz}\cos\theta+\Sigma_{xy}\cos\varphi\sin\theta+\Sigma_{yy}\sin\varphi\sin\theta\\ \Sigma_{zz}\cos\theta+\Sigma_{xz}\cos\varphi\sin\theta+\Sigma_{yz}\sin\varphi\sin\theta\end{pmatrix}_R$$

$$\underline{E}(t)=\begin{pmatrix}E_{xz}\cos\theta+E_{xx}\cos\varphi\sin\theta+E_{xy}\sin\varphi\sin\theta\\ E_{yz}\cos\theta+E_{xy}\cos\varphi\sin\theta+E_{yy}\sin\varphi\sin\theta\\ E_{zz}\cos\theta+E_{xz}\cos\varphi\sin\theta+E_{yz}\sin\varphi\sin\theta\end{pmatrix}_R$$

$$\underline{\Sigma}_{\underline{n}}(t)=\begin{bmatrix}\cos\varphi\sin\theta\left(\Sigma_{xz}\cos\theta+\Sigma_{xx}\cos\varphi\sin\theta+\Sigma_{xy}\sin\varphi\sin\theta\right)+\\ \sin\varphi\sin\theta\left(\Sigma_{yz}\cos\theta+\Sigma_{xy}\cos\varphi\sin\theta+\Sigma_{yy}\sin\varphi\sin\theta\right)+\cos\theta\left(\Sigma_{zz}\cos\theta+\Sigma_{xz}\cos\varphi\sin\theta+\Sigma_{yz}\sin\varphi\sin\theta\right)\end{bmatrix}\begin{pmatrix}\sin\theta\ \cos\varphi\\ \sin\theta\ \sin\varphi\\ \cos\theta\end{pmatrix}_R$$

The expression of $\underline{E}_{\underline{n}}(t)$ is obtained from the equation of $\underline{\Sigma}_{\underline{n}}(t)$ by changing Σ_{ij} in E_{ij}.
By using the shear stress vector definition: $\underline{T}(t) = \underline{\Sigma}_{\underline{n}}(t) - \underline{\sigma}(t)$, and the shear strain vector definition: $\underline{\gamma}(t) = \underline{E}_{\underline{n}}(t) - \underline{\varepsilon}(t)$, and the linear elastic behaviour of the material one can compute the following integral:

$$\int_{\varphi=0}^{2\pi}\underline{T}(t)\cdot\underline{Y}(t)\,d\varphi=\frac{\pi(1+\nu)}{4E}\begin{pmatrix}\frac{7}{8}\Sigma_{xx}^2+\frac{5}{2}\Sigma_{xy}^2+4\Sigma_{xz}^2-\frac{3}{4}\Sigma_{xx}\Sigma_{yy}+\frac{7}{8}\Sigma_{yy}^2+4\Sigma_{yz}^2-\Sigma_{xx}\Sigma_{zz}-\Sigma_{yy}\Sigma_{zz}+\Sigma_{zz}^2\\ +\left[-\frac{1}{2}\Sigma_{xx}^2-2\Sigma_{xy}^2+2\Sigma_{xz}^2+\pi\Sigma_{xx}\Sigma_{yy}-\frac{1}{2}\Sigma_{yy}^2+2\Sigma_{yz}^2\right]\cos2\theta\\ +\left[-\frac{3}{8}\Sigma_{xx}^2-\frac{1}{2}\Sigma_{xy}^2+2\Sigma_{xz}^2-\frac{1}{4}\Sigma_{xx}\Sigma_{yy}-\frac{3}{8}\Sigma_{yy}^2+2\Sigma_{yz}^2+\Sigma_{xx}\Sigma_{zz}+\Sigma_{yy}\Sigma_{zz}-\Sigma_{zz}^2\right]\cos4\theta\end{pmatrix}$$

Then,

$$\frac{1}{4\pi}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\underline{T}(t)\cdot\underline{Y}(t)\sin\theta d\theta d\varphi=\frac{2(1+\nu)}{15E}\left[\Sigma_{xx}^2+\Sigma_{yy}^2+\Sigma_{zz}^2+3\left(\Sigma_{xy}^2+\Sigma_{yz}^2+\Sigma_{xz}^2\right)-\left(\Sigma_{xx}\Sigma_{yy}+\Sigma_{xx}\Sigma_{zz}+\Sigma_{yy}\Sigma_{zz}\right)\right]$$

So, $\hat{\psi}_{\underline{n},Y} = \frac{2}{5}\Phi_d$.

For an isotropic material with a linear elastic behaviour it is easy to demonstrate that:

$$\underline{\sigma}(t)\cdot\underline{\varepsilon}(t)=\left(\frac{1+\nu}{E}\right)\sigma^2-\frac{3\nu}{E}\Sigma_H\sigma(t)$$

Then, since $\frac{1}{4\pi}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\underline{\sigma}(t)\sin\theta d\theta d\varphi=\Sigma_H$ and

$$\frac{1}{4\pi}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\underline{\sigma}^2(t)\sin\theta d\theta d\varphi=\frac{1}{15}\left[3\left(\Sigma_{xx}^2+\Sigma_{yy}^2+\Sigma_{zz}^2\right)+4\left(\Sigma_{xy}^2+\Sigma_{yz}^2+\Sigma_{xz}^2\right)+2\left(\Sigma_{xx}\Sigma_{yy}+\Sigma_{xx}\Sigma_{zz}+\Sigma_{yy}\Sigma_{zz}\right)\right]$$

$$\hat{\psi}_{\underline{n},\varepsilon}=\frac{1}{4\pi}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\underline{\sigma}(t)\cdot\underline{\varepsilon}(t)\sin\theta d\theta d\varphi$$

$$\Leftrightarrow\hat{\psi}_{\underline{n},\varepsilon}=\frac{(1+\nu)}{15E}\left[3\left(\Sigma_{xx}^2+\Sigma_{yy}^2+\Sigma_{zz}^2\right)+4\left(\Sigma_{xy}^2+\Sigma_{yz}^2+\Sigma_{xz}^2\right)+2\left(\Sigma_{xx}\Sigma_{yy}+\Sigma_{xx}\Sigma_{zz}+\Sigma_{yy}\Sigma_{zz}\right)\right]-\frac{3\nu}{E}\left(\Sigma_{xx}+\Sigma_{yy}+\Sigma_{zz}\right)^2$$

Thus, $\hat{\psi}_{\underline{n},\varepsilon}=\frac{2}{3}\left(\Phi_s+\frac{2}{5}\Phi_d\right).$

Annex 2: Calculation of $\frac{1}{T}\int_0^T \psi_{\underline{n},\Upsilon,a}(t)\,dt$**.** As explained in the paper, Papadopoulos demonstrated that in the frame $(O;\underline{u},\underline{v},\underline{w})$ for any synchronous multiaxial loading: $\Sigma_{ij}(t) = \Sigma_{ij,m} + \Sigma_{ij,a}\sin(\varpi t - \phi_{ij})$, the shear stress vector $\underline{\tau}(t)$. acting on the material plane Δ orientated by $\underline{n}$. can be written as

$\underline{T}(t) = T_u(t)\underline{u} + T_v(t)\underline{v}$ with $T_u(t) = A\sin\omega t + B\cos\omega t + T_{u,m}$ and $T_v(t) = C\sin\omega t + D\cos\omega t + T_{v,m}$

where A, B, C, and D are function of the phase difference ϕ_{ij} and of the stress amplitudes $\Sigma_{ij,a}$ [16]:

$$A = \mathrm{Sin}(\theta)\left(\left(\frac{-\Sigma_{xx,a}}{2}\mathrm{Cos}(\phi_{xx}) + \frac{\Sigma_{yy,a}}{2}\mathrm{Cos}(\phi_{yy})\right)\mathrm{Sin}(2\varphi) + \Sigma_{xy,a}\mathrm{Cos}(\phi_{xy})\mathrm{Cos}(2\varphi)\right) + \mathrm{Cos}(\theta)\left(-\Sigma_{xz,a}\mathrm{Cos}(\phi_{xz})\mathrm{Sin}(\varphi) + \Sigma_{yz,a}\mathrm{Cos}(\phi_{yz})\mathrm{Cos}(\varphi)\right)$$

$$B = \mathrm{Sin}(\theta)\left(\left(\frac{\Sigma_{xx,a}}{2}\mathrm{Sin}(\phi_{xx}) - \frac{\Sigma_{yy,a}}{2}\mathrm{Sin}(\phi_{yy})\right)\mathrm{Sin}(2\varphi) - \Sigma_{xy,a}\mathrm{Sin}(\phi_{xy})\mathrm{Cos}(2\varphi)\right) + \mathrm{Cos}(\theta)\left(\Sigma_{xz,a}\mathrm{Sin}(\phi_{xz})\mathrm{Sin}(\varphi) - \Sigma_{yz,a}\mathrm{Sin}(\phi_{yz})\mathrm{Cos}(\varphi)\right)$$

$$C = -\mathrm{Sin}(2\theta)\left(\frac{\Sigma_{xx,a}}{2}\mathrm{Cos}(\phi_{xx})\mathrm{Cos}^2(\varphi) + \frac{\Sigma_{yy,a}}{2}\mathrm{Cos}(\phi_{yy})\mathrm{Sin}^2(\varphi) - \frac{\Sigma_{zz,a}}{2}\mathrm{Cos}(\phi_{zz})\right) - \Sigma_{xy,a}\mathrm{Cos}(\phi_{xy})\mathrm{Sin}(2\theta)\mathrm{Sin}(2\varphi) - \mathrm{Cos}(2\theta)\left(\Sigma_{xz,a}\mathrm{Cos}(\phi_{xz})\mathrm{Cos}(\varphi) + \Sigma_{yz,a}\mathrm{Cos}(\phi_{yz})\mathrm{Sin}(\varphi)\right)$$

$$D = \mathrm{Sin}(2\theta)\left(\frac{\Sigma_{xx,a}}{2}\mathrm{Sin}(\phi_{xx})\mathrm{Cos}^2(\varphi) + \frac{\Sigma_{yy,a}}{2}\mathrm{Sin}(\phi_{yy})\mathrm{Sin}^2(\varphi) - \frac{\Sigma_{zz,a}}{2}\mathrm{Sin}(\phi_{zz})\right) + \frac{\Sigma_{xy,a}}{2}\mathrm{Sin}(\phi_{xy})\mathrm{Sin}(2\theta)\mathrm{Sin}(2\varphi) + \mathrm{Cos}(2\theta)\left(\Sigma_{xz,a}\mathrm{Sin}(\phi_{xz})\mathrm{Cos}(\varphi) + \Sigma_{yz,a}\mathrm{Sin}(\phi_{yz})\mathrm{Sin}(\varphi)\right)$$

In the frame $(O;\underline{u},\underline{v},\underline{w})$, $T_{u,m}$ and $T_{v,m}$ are the coordinates of the mean (or static) shear stress vector; $(A\sin\omega t + B\cos\omega t)$ and $(C\sin\omega t + D\cos\omega t)$ are the coordinates of the dynamic (or alternated) part of the shear stress vector.

From these relations the scalar product $\underline{T}(t)\cdot\underline{T}(t)$ is: $\underline{T}(t)\cdot\underline{T}(t) = T_u(t)^2 + T_v(t)^2$ with $T_u(t) = A\sin\omega t + B\cos\omega t + T_{u,m}$ and $T_v(t) = C\sin\omega t + D\cos\omega t + T_{v,m}$.

The integration over a cycle leads to: $\frac{1}{T}\int_0^T T_u(t)^2 dt = \frac{A^2}{2} + \frac{B^2}{2} + T_{u,m}{}^2$ and $\frac{1}{T}\int_0^T T_v(t)^2 dt = \frac{C^2}{2} + \frac{D^2}{2} + T_{v,m}{}^2$

Finally, $\frac{1}{T}\int_0^T \underline{T}(t)\cdot\underline{T}(t)dt = \frac{1}{T}\int_0^T T_u(t)^2 dt + \frac{1}{T}\int_0^T T_v(t)^2 dt = \frac{A^2}{2} + \frac{B^2}{2} + \frac{C^2}{2} + \frac{D^2}{2} + T_{u,m}{}^2 + T_{v,m}{}^2$.

The last two terms $(T_{u,m}{}^2 + T_{v,m}{}^2)$ represent to the mean value of the shear stress vector. In other words, they are due to the static part of the loading, while $\left(\frac{A^2}{2} + \frac{B^2}{2} + \frac{C^2}{2} + \frac{D^2}{2}\right)$ corresponds to the dynamic part of the loading.

Moreover, for an isotropic material with an elastic linear behaviour it comes:

$$\underline{Y}(t) = \frac{1+\nu}{E}\underline{T}(t)$$

So, $\frac{1}{T}\int_0^T \psi_{\underline{n},\gamma,a}(t)dt = \frac{1}{T}\int_0^T \underline{T}_a(t)\cdot\underline{\gamma}_a(t)dt = \frac{1+\nu}{E}\left(\frac{A^2+B^2+C^2+D^2}{2}\right)$.

Annex 3: Calculation of the shear work given on a plane. For an homogeneous isotropic linear elastic material the amplitudes of the shear stress and the shear strain vectors are related by: $\underline{\gamma}_a(t) = \frac{1+\nu}{E}\underline{\tau}_a(t)$.

From Annex 2 it is easy to prove that: $\underline{\tau}_a(t)\cdot\dot{\underline{\gamma}}_a(t) = \frac{1+\nu}{E}\left[\tau_{u,a}(t)\dot{\tau}_{u,a}(t) + \tau_{v,a}(t)\dot{\tau}_{v,a}(t)\right]$ with $T_{u,a}(t) = A\sin\omega t + B\cos\omega t$ and $T_{v,a}(t) = C\sin\omega t + D\cos\omega t$.

Thus, $\underline{\tau}_a(t)\cdot\dot{\underline{\gamma}}_a(t) = \left(\frac{1+\nu}{E}\right)\omega\left[G\sin(2\omega t) + H\cos(2\omega t)\right]$ with $G = (A^2 - B^2 + C^2 - D^2)/2$ and $H = (AB + CD)$.

So $\underline{\tau}_a(t)\cdot\dot{\underline{\gamma}}_a(t) = \left(\frac{1+\nu}{E}\right)\omega\sqrt{G^2+H^2}\sin\left[2\omega t + \mathrm{Arctan}(H/G)\right]$. From this last equation it comes:

$$Wg_{\underline{n},\gamma,a} = \int_T \left\langle 2\underline{\tau}_a(t)\cdot\dot{\underline{\gamma}}_a(t)\right\rangle dt = \frac{4(1+\nu)}{E}\sqrt{G^2+H^2} .$$

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Abstract

Kritisches ebenes Konzept und Energieansatz für mehrachsige Ermüdung. Der vorliegende Beitrag zielt auf eine neue Interpretation verschiedener energiebasierender ebener Kriterien für die Ermüdung bei hohen Lastwechseln. Genauer gesagt wird die Bedeutung verschiedener globaler ebener energiebasierender Kriterien aus Berechnungen der Normal- und Scherbeanspruchung einer kritischen Ebene detailliert. Die Betrachtung ist begrenzt auf die mehrachsige Ermüdung bei konstanter Amplitude und synchroner proportionaler und nicht proportionaler Beanspruchung, mit und ohne Hauptbeanspruchung. Nach einem kurzen Überblick einiger energiebasierender Kriterien unter Nutzung des Konzepts der kritischen Ebene oder eines globalen Ansatzes werden die Gründe dafür, warum die energiebasierenden Kriterien kritischer Ebenen in der Literatur immer weiter entwickelt werden, aufgezeigt. Es wird gezeigt, dass eine direkte Verbindung zwischen den Parametern der Energie kritischer Ebenen und globalen Parametern existiert. Es wird eine kritische Untersuchung der berühmten Experimente von Finley et al. vorgenommen und die angesprochene Verbindung wird anhand zweier von Papadopoulos entwickelten mesoskopischen Kriterien und dem auf der globalen Energie basierten Kriterium nach Banvillet et al. Beleuchtet. Dies führt zur Annahme, dass die Initiierung von Ermüdungsrissen von der mesoskopischen bis zur makroskopischen Betrachtung entweder mittels Spannungs- oder Dehnungsteilen oder mittels verformungsenergetischen Parametern interpretiert werden kann.

The Authors of This Contribution

Dr. Thierry Palin-Luc was born in 1968. After an engineering diploma of E.N.S.A.M. obtained in 1991, he prepared a PhD thesis at LAMEFIP about multiaxial fatigue of spheroidal graphite cast iron under combined loadings (defended in 1996). Since 1997, he is assistant professor in this laboratory. He works on multiaxial fatigue of materials and structures and developped volumetric energy based multiaxial fatigue models.

Dr. Franck Morel was born in 1967. After an engineering diploma of C.E.S.T.I. (SupMéca) obtained in 1991, he prepared a PhD thesis at Poitiers (LMPM) about multiaxial loading under variable amplitude loading (defended in 1996). Since 1997, he is assistant professor first in poitiers and now in Angers (ENSAM). His main fields of research are fatigue under complex loading and the influence of the material processing on the fatigue properties.