Quasi-trivial stacking sequences for the design of thick laminates

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A B S T R A C T

Quasi-trivial (QT) sequences have largely proven to be an extremely powerful tool in the design and optimisation of composites laminates. In this paper new interesting properties of this class of stacks are derived. These properties allow to obtain QT sequences by superposing (according to some prescribed rules) any number of QT elementary stacks. In this way, QT solutions with arbitrary large number of plies can be readily obtained, overcoming the computational issues arising in the search of QT solutions with huge number of layers. Moreover, a general version of the combinatorial algorithm to find QT stacks is proposed in this work. It is also proven that the previous estimation of the number of QT solutions, for a given number of plies and saturated groups, is not correct because a larger number of solutions has been found in this study.

1. Introduction

The utilisation of composite materials has undergone a boost in recent years. Indeed, composites allow for a great range of design, over multiple and very different applications. However, the design of a composite structure is a complicated task because of anisotropy and heterogeneity of these materials. Heterogeneity mainly affects the behaviour of the material at the microscopic scale (i.e. that of the constitutive phases), while anisotropy essentially appears at mesoscopic (ply-level) and macroscopic (laminate-level) scales. When dealing with the design problem of composite structures, laminates with *identical* plies (i.e, laminates composed of constitutive plies having same material properties and thickness) are often used (e.g. in aeronautical and automotive applications). In this case, the variables that can be used to tailor the properties of the structure are the total number of plies and their orientation angle. Therefore, the simultaneous design of both structure geometry and laminate stack is of paramount importance. In this background, engineers make a systematic use of some simplifying hypotheses/rules to get some desired properties (membrane/bending uncoupling, membrane orthotropy, etc.), which are difficult to be mathematically formalised and hard to be obtained otherwise. Unluckily, these design rules (e.g. symmetric stacks to get membrane/bending uncoupling, balanced ones to get membrane orthotropy, etc.) drastically reduce the design space and often lead to cut out entire classes of stacks that could potentially represent optimum solutions for the problem at hand.

In this context, the introduction of Quasi-Trivial (QT) stacking sequences in 2001 by Vannucci and Verchery [1] represented a major improvement. In [1], the authors utilised the polar formalism [2], in the framework of the Classical Laminate Theory (CLT), to derive the equations defining the general conditions for membrane-bending uncoupling and *quasi-homogeneity* (i.e. uncoupled laminates with same behaviour in terms of normalised membrane and bending stiffness tensors) for a laminate made of identical plies. Indeed, QT stacking sequences are a class of exact solutions to these equations.

Since their derivation, QT solutions have been used for many scopes, mainly in the field of laminates design and optimisation. In [3] the authors analysed the problem of superposing laminates by means of the polar formalism and inferred that QT solutions are not the only ones satisfying the requirements of uncoupling and quasi-homogeneity. Nevertheless, they did not go further in the analysis of superposed QT solutions and simply concluded that, generally speaking, the superposition of two QT stacks does not give rise to a QT one. In [4], QT solutions have been used together with the well known Werren and Norris rule to obtain fully isotropic laminates. In [5] anti-symmetrical uncoupled stacking sequences have been used to obtain fully orthotropic laminates. In [6] York proposed a list of fully orthotropic angle-ply laminates and some rules to mix these sequences. In [7] the same author proposed a list of fully uncoupled extensionally isotropic, fully isotropic and quasi homogeneous angle-ply laminates. Despite these list is said to be definitive, counter examples were found, suggesting this is not currently the case.

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In [8], Vannucci et al. proved that one can obtain fully orthotropic laminates by simply using QT quasi-homogeneous stacks with angle-ply orientations. They used these sequences for searching optimum flexural solutions. Jibawy et al. [9] made use of the same idea within an optimisation procedure in order to constrain the solutions to be quasi-homogeneous orthotropic ones. In [10] Montemurro and Catapano utilised QT quasi-homogeneous stacks in the framework of the multi-scale two-level optimisation of variable angle tow laminates.

In this paper, simple, general and extremely useful rules to obtain QT stacks by superposition of elementary QT sequences are derived. Here, an *elementary* QT stack must be interpreted as a QT solution with a low number of plies that can be obtained by means of the combinatorial algorithm whose general architecture is also presented in this study. These rules are derived in a very general way, i.e. they apply for:

1. any number q of elementary QT solutions to be superposed;
2. any number n_i of plies of each i-th elementary QT sequence;
3. any number of orientation groups in each elementary QT solution.

They allow generating QT stacking sequences with an arbitrarily high number of plies. For this reason, they are very important in the framework of the design/optimisation of thick laminates or laminates composed of a huge number of thin plies [11]. This achievement represents a major improvement, because the search for QT solutions is limited by computational costs as the number of plies composing the stack increases. Indeed, up to now only sequences with a low number of plies have been found [1].

The rest of the paper is organised as follows: Section 2 recalls the fundamentals of QT solutions while Section 3 summarises the numerical procedure to build the database of QT solutions. Section 4 introduces the mathematical formalisation of the problem of finding QT solutions by superposition of elementary QT sequences. Sections 5–7 report the rules to obtain QT sequences by superposition for the case of uncoupled, homogeneous and quasi-homogeneous laminates, respectively. In Section 8 the properties of solutions resulting from the application of the previous rules are validated through meaningful numerical examples. Finally, Section 9 ends the paper with some concluding remarks.

2. Fundamentals and properties of quasi-trivial solutions

Consider a multilayer plate composed of n plies as illustrated in Fig. 1. Axes x and y are assumed to be on the laminate middle plane, while axis z is perpendicular to this plane. The CLT gives the constitutive relationship (using Voigt's notation) between generalised forces (i.e. forces and moments per unit length) and generalised strains (strains and curvatures) of the middle plane:

$$\mathbf{N} = \mathbf{A}\,\epsilon_0 + \mathbf{B}\chi,\quad \mathbf{M} = \mathbf{B}\,\epsilon_0 + \mathbf{D}\chi. \tag{1}$$

In Eq. (1), $\mathbf{N}$ is the vector of in-plane resultant forces per unit length,

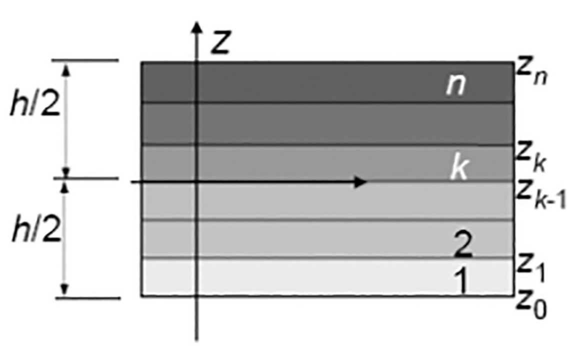

Fig. 1. Laminate stack parameters and notation.

$\mathbf{M}$ is the vector of bending moments, ϵ_0 is the vector of in-plane strains of the middle plane of the laminate while χ is the curvatures vector. $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$ are the membrane, membrane/bending coupling and bending stiffness matrices, respectively. For a laminate with identical plies the expressions of these matrices in terms of the geometrical and material parameters of the stack are:

$$\mathbf{A}= \frac{h}{n}\sum_{k=1}^{n} \mathbf{Q}_k(\delta_k),\quad \mathbf{B}= \frac{1}{2}\frac{h^2}{n^2}\sum_{k=1}^{n} b_k\mathbf{Q}_k(\delta_k),\quad \mathbf{D}= \frac{1}{12}\frac{h^3}{n^3}\sum_{k=1}^{n} d_k\mathbf{Q}_k(\delta_k). \tag{2}$$

In Eq. (2) $\mathbf{Q}_k$ is the reduced stiffness matrix of the k-th constitutive ply, while δ_k is its orientation angle. Coefficients b_k and d_k depend on the position k of the ply within the stack:

$$b_k = 2k-n-1, \tag{3}$$

$$d_k = 12k(k-n-1) + 4 + 3n(n + 2). \tag{4}$$

For convenience, the normalised stiffness matrices are defined as follows:

$$\mathbf{A}^* = \frac{\mathbf{A}}{h},\quad \mathbf{B}^* = 2\frac{\mathbf{B}}{h^2},\quad \mathbf{D}^* = 12\frac{\mathbf{D}}{h^3}. \tag{5}$$

These matrices have all the same units and can thus be used for comparison purposes. In addition, it is possible to define the laminate homogeneity stiffness matrix:

$$\mathbf{C} = \mathbf{A}^*-\mathbf{D}^*, \tag{6}$$

which represents a measure of the difference between the normalised membrane and bending behaviours.

A laminate is said to be uncoupled if:

$$\mathbf{B}^* = \mathbf{0}, \tag{7}$$

while it is said homogeneous if:

$$\mathbf{C} = \mathbf{0}. \tag{8}$$

Finally a laminate is *quasi-homogeneous* if properties (7) and (8) hold simultaneously.

Vannucci and Verchery [1] made use of the polar formalism to represent matrices $\mathbf{A}^*$, $\mathbf{B}^*$, $\mathbf{D}^*$ and $\mathbf{C}$. They proved that, for the case of a laminate composed of identical plies, the isotropic part of stiffness matrices $\mathbf{B}^*$ and $\mathbf{C}$ automatically satisfies the above equations and only the anisotropic part is thus relevant to the problem. The conditions for uncoupling and homogeneity (and hence for quasi-homogeneity) can be resumed as follows (see [1] for more details):

$$\sum_{k=1}^{n} b_k e^{4i\delta_k} = 0,\quad \sum_{k=1}^{n} b_k e^{2i\delta_k} = 0, \tag{9}$$

and

$$\sum_{k=1}^{n} c_k e^{4i\delta_k} = 0,\quad \sum_{k=1}^{n} c_k e^{2i\delta_k} = 0, \tag{10}$$

where c_k is a coefficient related to matrix $\mathbf{C}$ whose expression is:

$$c_k = -2n^2-12k(k-n-1)-4-6n. \tag{11}$$

As discussed in [1], one can observe that coefficients b_k and c_k have some interesting properties: b_k varies linearly with the position k of the ply, whilst c_k is symmetric with a parabolic variation with respect to k, see Fig. 2. In addition, the sum of each coefficient over the interval $[1, n]$ is always null,

$$\sum_{k=1}^{n} b_k = 0,\quad \sum_{k=1}^{n} c_k = 0. \tag{12}$$

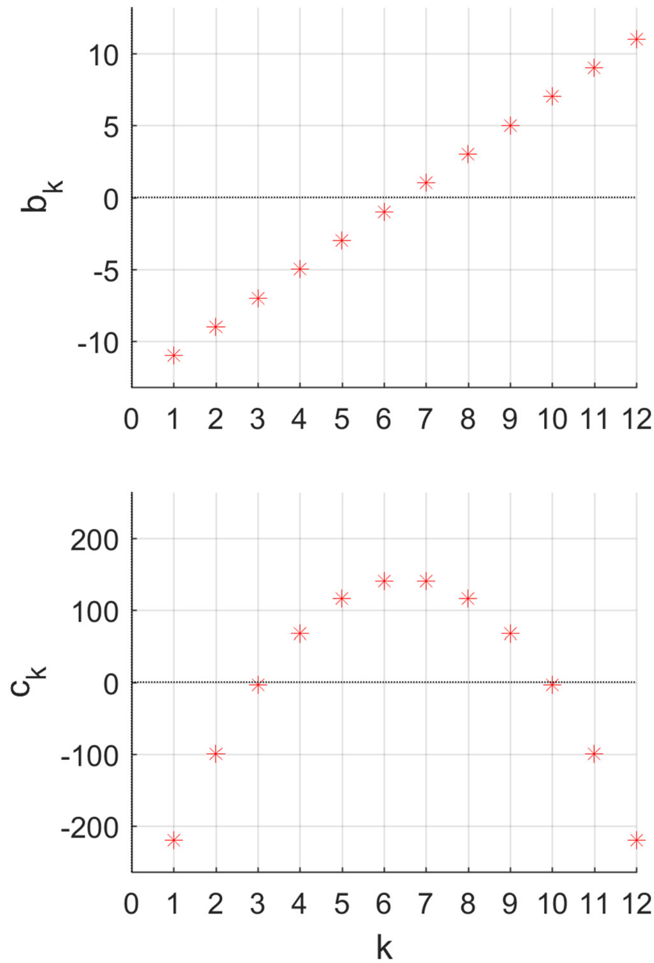

Fig. 2. Coefficients b_k and c_k as functions of k for a 12 plies laminate.

To explain clearly the concept of QT solutions, consider a laminate which is composed of n plies and only m different orientation angles, with $m \leqslant n$. Let G_j be the set of plies sharing the same orientation angle θ_j, i.e.

$$G_j = \{k: \delta_k = \theta_j\}. \tag{13}$$

Of course, the union of these sets gives the full set of position indexes of the laminate, namely $k = 1, \ldots, n$.

Each of expressions in Eqs. (9) and (10) can be split as a multiple sum over the different sets $G_j, j = 1, \ldots, m$:

$$\sum_{k=1}^{n} b_k e^{4i\delta_k} = \sum_{j=1}^{m} e^{4i\theta_j} \sum_{k\in G_j} b_k,$$
$$\sum_{k=1}^{n} b_k e^{2i\delta_k} = \sum_{j=1}^{m} e^{2i\theta_j} \sum_{k\in G_j} b_k, \tag{14}$$

$$\sum_{k=1}^{n} c_k e^{4i\delta_k} = \sum_{j=1}^{m} e^{4i\theta_j} \sum_{k\in G_j} c_k,$$
$$\sum_{k=1}^{n} c_k e^{2i\delta_k} = \sum_{j=1}^{m} e^{2i\theta_j} \sum_{k\in G_j} c_k. \tag{15}$$

From these expressions it is evident that if the sum of coefficients b_k (or c_k) is null over each set G_j, then uncoupling and/or homogeneity requirements are satisfied, regardless the value of the orientation angle in each group. In this context, a group of plies oriented at θ_j, for which:

$$\sum_{k\in G_j} b_k = 0, \quad j = 1, \ldots, m, \tag{16}$$

$$\sum_{k\in G_j} c_k = 0 \quad j = 1, \ldots, m, \tag{17}$$

is called *saturated group* with respect to coefficients b_k or c_k, respectively, and the related set of indexes G_j is called *saturated set*. A QT stack is entirely composed of saturated groups.

It is noteworthy that a QT stack can satisfy uncoupling, homogeneity or quasi-homogeneity conditions regardless to the value of the orientation angle characterising each saturated group, i.e. θ_j, which can get any value without altering the nature of the stack. As a matter of fact, the orientation angles of saturated groups constitute additional free design variables that can be properly chosen/optimised to satisfy further requirements (elastic properties along some prescribed directions, buckling behaviour, natural frequencies, etc.).

3. Creation of the quasi-trivial database

A complete database of QT solutions, for each combination of n and m, has been obtained by means of a dedicated algorithm. The basic structure of such an algorithm, entitled *QT stacks finder*, is represented here below.

Algorithm 1 QT stacks finder

1. Set n and $m = 2$, $n > m$
2. Search for QT solutions
   2.1 ∀ combination of the number of layers belonging to each G_j $(j = 1, \ldots, m)$
   2.2 ∀ permutation of $k \in G_j$ of a given combination
      2.2.1 find sequences for which Eq. (9) is verified (uncoupled QT solutions)
      2.2.2 find sequences for which Eq. (10) is verified (homogeneous QT solutions)
      2.2.3 select sequences resulting from steps 2.2.1 and 2.2.2 and look for those meeting Eqs. (9) and (10) simultaneously (quasi-homogeneous QT solutions).
3. Classify QT solutions found in step 2 in three main groups
   3.1 QT independent solutions (see [1]) that are saved in the database
   3.2 QT potentially "growing" solutions (solutions that can be modified to give rise to QT solutions characterised by higher values of m) that will be utilised in Step 4
   3.3 QT dependent solutions that are deleted
4. ∀ QT solution of step 3.2 set $m = m + 1$ and repeat steps from 2 to 4 until the stopping criterion is met

The concept behind Algorithm 1 is quite simple. The user must set the number of plies n and the algorithm starts to search for QT solutions with $m = 2$ satisfying Eq. (9) and/or (10), according to the problem at hand. Once QT stacks have been found, they are grouped into three different categories: independent QT solutions (according to the definition given in [3]), dependent QT solutions (which are deleted from the database) and *potentially growing* QT solutions. This last class of QT sequences is composed of "special" dependent QT solutions showing an interesting property. Indeed, among the different saturated sets $G_j, j = 1, \ldots, m$ characterising the generic solution belonging to this class, it exists at least one subset of plies indexes, i.e. $k \in S_j \subset G_j$, that is still a saturated set. In this case, such a QT solution can become independent if a new orientation group ($m = m + 1$) is substituted in place of the existing saturated subset S_j. After this substitution, the algorithm continue to search for QT solutions for $m = m + 1$ until the stopping criterion is met, i.e. when no potentially growing QT solutions can be identified, thus no further incrementation of the number of orientation groups m is possible.

The number of QT solutions for the cases of membrane/bending uncoupling, homogeneity and quasi-homogeneity is listed in Tables

Table 1
Number of independent QT uncoupled solutions obtained as a function of total number of plies and number of orientation groups.

N. of plies n	N. of groups m										N. of solutions
	2	3	4	5	6	7	8	9	10	11	12	
7	0	1	1	0	0	0	0	0	0	0	0	2
8	1	0	1	0	0	0	0	0	0	0	0	2
9	0	1	2	1	0	0	0	0	0	0	0	4
10	0	4	0	1	0	0	0	0	0	0	0	5
11	0	0	9	4	1	0	0	0	0	0	0	14
12	1	8	9	0	1	0	0	0	0	0	0	19
13	0	0	25	32	6	1	0	0	0	0	0	64
14	0	37	34	17	0	1	0	0	0	0	0	89
15	0	0	10	207	78	9	1	0	0	0	0	305
16	0	58	305	96	29	0	1	0	0	0	0	489
17	0	0	2	893	895	144	12	1	0	0	0	1947
18	0	114	1492	1262	208	45	0	1	0	0	0	3122
19	0	0	0	2216	8192	2663	264	16	1	0	0	13352
20	0	0	7391	11240	3683	396	66	0	1	0	0	22777
21	0	0	0	4936	59701	39986	6283	406	20	1	0	111333
22	0	0	29144	101207	49008	8869	694	93	0	1	0	189016
23	0	0	0	6369	346057	519231	141298	13130	626	25	1	1026737
24	0	0	75421	844224	665507	156300	18569	1118	126	0	1	1761266

Table 2
Number of independent QT solutions with $\mathbf{C} = \mathbf{0}$ obtained as a function of total number of plies and number of orientation groups.

N. of plies n	N. of groups m					N. of solutions
	2	3	4	5	6	
4	2	0	0	0	0	2
5	2	0	0	0	0	2
6	4	0	0	0	0	4
7	0	1	0	0	0	1
8	8	0	0	0	0	8
9	4	0	0	0	0	4
10	20	8	0	0	0	28
11	0	22	0	0	0	22
12	36	0	0	0	0	36
13	16	52	0	0	0	68
14	2	12	32	128	16	190
15	0	100	0	0 0	0	100
16	0	32	40	16	32	120
17	142	652	32	0	0	826
18	34	720	336	16	0	1106
19	4	1436	4232	512	0	6184
20	68	4856	5104	0	0	10028
21	26	500	1168	1248	0	2942
22	0	36804	302832	139424	4864	483924
23	50	164918	129212	2016	0	296196
24	152	5864	159632	0	0	165648
25	0	314018	665512	123044	4000	1106574

Table 3
Number of independent QT quasi homogeneous solutions obtained as a function of total number of plies and number of orientation groups; symmetric solutions are reported in parentheses.

N. of plies n	N. of groups m					N. of solutions
	2	3	4	5	6	
7	1(1)	0	0	0	0	1(1)
8	1	0	0	0	0	1
9	0	0	0	0	0	0
10	0	0	0	0	0	0
11	4(2)	0	0	0	0	4(2)
12	1	0	0	0	0	1
13	4	3	0	0	0	7
14	0	2(1)	0	0	0	2(1)
15	4	3	0	0	0	7
16	6	3(1)	0	0	0	9(1)
17	30	11	0	0	0	41
18	0	9	0	0	0	9
19	60	41	0	0	0	101
20	52	17	1	0	0	70
21	62	18(2)	0	0	0	80(2)
22	32(2)	188(1)	26	2	0	248(3)
23	189(1)	970	0	0	0	1159(1)
24	248	47	1	0	0	296
25	326	4184	98	0	0	4608
26	108	2065 (2)	672 (3)	41 (2)	3	2889 (7)
27	171(1)	1804	510	39	1	2525(1)
28	357	9492(1)	1691(2)	61	9	11610(3)
29	122	75281	15068	167	0	90638
30	106	10923	1009(3)	51	0	12089(3)
31	28	290227	156565(1)	1728	1	448549(1)
32	263	161436(5)	70091	4521	100	236411(5)
33	316	260442	112324	937	0	374019
34	716	1389039 (107)	568492 (35)	12589	38	1970874 (142)
35	2	8291650 (8)	6392064 (7)	90433	82	14774231 (15)

1–3, respectively. The proposed algorithm is able to find a higher number of QT solutions than those found in the past, see [3]. Nevertheless, when considering stacking sequences with a significant number of layers, i.e. $n \geqslant 35$, computational issues related to lack of memory and/or excessive computational time arise.

A quick glance to Tables 1,2 suffices to infer that the number of QT solutions presented in [3] is underestimated. For example, in [3] the number of QT quasi-homogeneous stacks obtained for $n = 13$ and $m = 3$ is two, while Algorithm 1 is able to determine three independent QT solutions, namely:

$$I = [1\ 2\ 3\ 1\ 1\ 3\ 1\ 2\ 1\ 1\ 2\ 3\ 1], \tag{18}$$

$$II = [1\ 2\ 3\ 1\ 3\ 1\ 3\ 2\ 1\ 1\ 2\ 1\ 3], \tag{19}$$

$$III = [1\ 2\ 3\ 2\ 2\ 3\ 1\ 2\ 1\ 2\ 1\ 3\ 2]. \tag{20}$$

One can immediately verify that these solutions are really independent, distinct and quasi-homogeneous ones. Table 4 lists the values of coefficient c_k for all ply indexes belonging to each saturated set. In particular, it can be noticed that no saturated subsets can be defined, hence no further growth of m is possible.

It is noteworthy that Tables 1–3 report the number of independent solutions only. This means that the overall number of QT solutions is considerably higher because also dependent solutions can be utilised for design purposes. To understand this concept, one can consider the first studied case reported in Table 1, i.e. when the *QT stack finder* algorithm

Table 4
Coefficient c_k for each saturated group of QT quasi-homogeneous stacking sequences with $n = 13$.

		Sequence												
I	j	1	2	3	1	1	3	1	2	1	1	2	3	1
	c_k	−264	−132	−24	60	120	156	168	156	120	60	−24	−132	−264
II	j	1	2	3	1	3	1	3	2	1	1	2	1	3
	c_k	−264	−132	−24	60	120	156	168	156	120	60	−24	−132	−264
III	j	1	2	3	2	2	3	1	2	1	2	1	3	2
	c_k	−264	−132	−24	60	120	156	168	156	120	60	−24	−132	−264

is utilised to search for QT uncoupled solutions with seven plies. A quick glance to Table 1 suffices to verify that, in this case, only two independent solutions exist: with three and four saturated group respectively. However, the real number of QT solutions is higher. When Algorithm 1 starts the search of QT solutions with $m = 2$, it finds only growing-up QT solutions (according to the definition given before). Such solutions are not reported in Table 1 because they represent a special case of QT independent solutions with three and four orientation groups, respectively. Finally, for $n = 7$ and $m = 4$ Algorithm 1 finds no more potentially growing QT solutions, thus the stopping criterion is met and a new solution search can be started in the family of QT solutions with eight plies.

4. How to get a quasi-trivial stack by superposing quasi-trivial elementary stacks?

As discussed in the previous Section, the utilisation of Algorithm 1 needs important computational resources when the number of plies is greater than 35, i.e. when considering moderately thick or thick laminates. From an engineering point of view, it is really important to determine some general rules which allow getting QT stacking sequences made of a large number of layers by superposing QT elementary stacks (characterised by a lower number of plies). These rules are of paramount importance when dealing with the problem of designing/optimising thick composite structures showing particular elastic properties (e.g. quasi-homogeneity) which cannot be obtained with standard rules (symmetric stacks, balanced stacks, etc.). However, the stack resulting from a simple superposition of two (or more) QT solutions is not necessarily a QT one. As a matter of fact, when superposing two QT solutions in the resulting sequence the ply indexes k belonging to each saturated set G_j are shifted, thus Eqs. (16) and (17) could not apply for each set. Therefore, a criterion must be defined to ensure that the resulting stack is still a QT solution.

In the following Section criteria to obtain a QT sequence by superposing a given number of QT elementary solutions are derived. To do so, an appropriate notation must be introduced.

First of all, QT solutions to be superposed are called *initial* or *elementary* QT stacks, while that resulting from this superposition is called QT *macro-sequence* or *macro-stack*. The reference of Fig. 1 will be used for both elementary QT stacks and QT macro-stacks.

Consider the superposition of q QT sequences and refer to the scheme illustrated in Fig. 3. The number of plies of the i-th initial QT solution is denoted by n_i. Therefore the number of plies of the QT macro-sequence is:

$$n_{tot} = \sum_{i=1}^{q} n_i. \tag{21}$$

According to Fig. 3, QT elementary solutions are superposed in bottom-to-top order. The first elementary solution QT_1 is placed at the bottom of the QT macro-stack while the last one, QT_q, is placed at the top. To refer to quantities related to the QT macro-sequence the symbol $*$ is added as a superscript. For instance, in the QT macro-sequence the position of each ply is denoted by k^* index, while index k stands for the position of layers in the elementary QT solution. Let $K^{(i)}$ be the set of k belonging to elementary i-th stack QT_i. According to Fig. 3, the relationship between k^* and k is:

$$k^* = \begin{cases} k & \text{if } k \in K^{(1)}, \\ k + \sum_{r=1}^{i-1} n_r & \text{if } k \in K^{(i)}, \quad i = 2, \dots, q. \end{cases} \tag{22}$$

By introducing the following quantity,

$$\Delta k^{(i)} = \begin{cases} 0 & \text{if } i = 1, \\ \sum_{r=1}^{i-1} n_r & \text{if } i = 2, \dots, q, \end{cases} \tag{23}$$

Eq. (22) can be rewritten as

$$k^* = k + \Delta k^{(i)} \quad k \in K^{(i)}, \quad i = 1, \dots, q. \tag{24}$$

Index j will be used to denote a group of plies sharing the orientation θ_j, while $G_j^{(i)}$ will be used to refer to the set of k indexes belonging to the sequence QT_i and sharing the orientation θ_j. Similarly, G_j^* refers to the set of k^* indexes of plies belonging to the macro-sequence and oriented at θ_j. According to this notation, the following facts can be easily inferred.

1. Consider the i-th QT elementary solution and let m_i be the number of saturated groups composing QT_i. It follows that:

$$K^{(i)} = \bigcup_{j=1}^{m_i} G_j^{(i)}. \tag{25}$$

2. Given the orientation θ_j the set of G_j^* can be expressed as the union of saturated sets belonging to each QT elementary stack whose indexes has been expressed within the frame of the macro-stack, namely:

$$G_j^* = \bigcup_{r=1}^{q} G_j^{*,(r)}. \tag{26}$$

Finally, for a given pair of k and $\Delta k^{(i)}$ the following relationships apply:

$$b_{k^*} = b_{k+\Delta k^{(i)}}, \tag{27}$$

$$c_{k^*} = c_{k+\Delta k^{(i)}}, \tag{28}$$

$$\sum_{k^* \in G_j^{*,(i)}} b_{k^*} = \sum_{k \in G_j^{(i)}} b_{k+\Delta k^{(i)}}, \tag{29}$$

$$\sum_{k^* \in G_j^{*,(i)}} c_{k^*} = \sum_{k \in G_j^{(i)}} c_{k+\Delta k^{(i)}}. \tag{30}$$

5. Uncoupling of superposed QT stacking sequences

In this section an analytical rule is derived to obtain QT uncoupled macro-sequences by superposition of elementary QT uncoupled sequences.

In this regard, for the initial r-th sequence QT_r, $r = 1, \dots, q$, Eq. (16) stands and it can be written for each sequence as follows:

$$\sum_{k \in G_j^{(r)}} b_k = 0, \quad \begin{cases} \forall \; r = 1, \dots, q, \\ \forall \; j = 1, \dots, m_r, \end{cases} \tag{31}$$

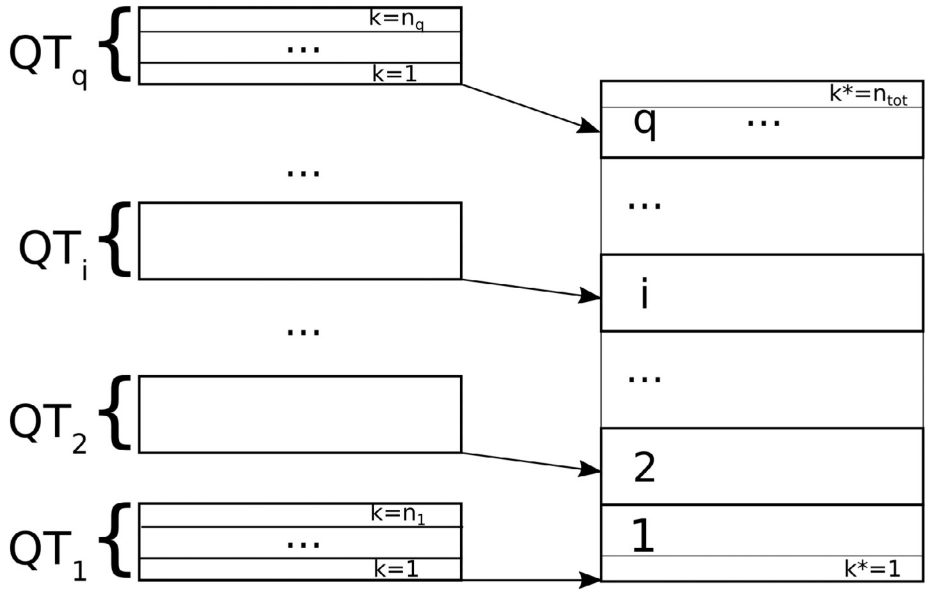

Fig. 3. General scheme of superposition of q QT elementary stacks.

where m_r represents the number of orientation groups belonging to sequence QT_r. Taking into account the general expression of b_k given in Eq. (3), Eq. (31) can be rewritten as follows:

$$\sum_{k\in G_j^{(r)}} 2k - n_r - 1 = 0, \quad \begin{cases} \forall \ r = 1, \dots, q, \\ \forall \ j = 1, \dots, m_r. \end{cases} \tag{32}$$

The macro-stack is a QT one if each orientation group is a saturated one in terms of the coefficient b_{k^*}, i.e. if each group within the macro-stack satisfies Eq. (16). This requirement can be formalised as:

$$\sum_{k^*\in G_j^*} b_{k^*} = 0, \quad j = 1, \dots, m^*, \tag{33}$$

where m^* represents the total number of orientations groups in the macro-sequence.

Considering Eq. (26), the sum over G_j^* in Eq. (33) can be split in multiple sum over $G_j^{*,(r)}$:

$$\sum_{k^*\in G_j^*} b_{k^*} = \sum_{r=1}^{q} \sum_{k^*\in G_j^{*,(r)}} b_{k^*} = 0, \quad j = 1, \dots, m^*. \tag{34}$$

Then, using Eq. (29) the previous relationship writes:

$$\sum_{r=1}^{q} \sum_{k^*\in G_j^{*,(r)}} b_{k^*} = \sum_{r=1}^{q} \sum_{k\in G_j^{(r)}} b_{k+\Delta k^{(r)}} = 0, \quad j = 1, \dots, m^*. \tag{35}$$

Then, replacing Eqs. (3) (carefully adapted to the notation of the QT macro-sequence) and (24) in Eq. (35), one obtains:

$$\sum_{r=1}^{q} \sum_{k\in G_j^{(r)}} [2(k + \Delta k^{(r)}) - n_{tot} - 1] = 0, \quad j = 1, \dots, m^*. \tag{36}$$

Eq. (36) can be further simplified by taking into account Eq. (32):

$$\sum_{r=1}^{q} n_{G_j^{(r)}} [2\Delta k^{(r)} + n_r - n_{tot}] = 0 \quad j = 1, \dots, m^*. \tag{37}$$

In Eq. (37) $n_{G_j^{(r)}}$ indicates the number of plies oriented at θ_j in the r-th elementary QT stack. Consider now Eqs. (21) and (23). The following equality holds:

$$n_r - n_{tot} = -\left(\sum_{i=1}^{r-1} n_i + \sum_{i=r+1}^{q} n_i\right) = -\left(\Delta k^{(r)} + \sum_{i=r+1}^{q} n_i\right), \tag{38}$$

By replacing Eq. (38) into Eq. (37) it is possible to get the desired criterion, namely:

$$\sum_{r=1}^{q} n_{G_j^{(r)}} \left[\Delta k^{(r)} - \sum_{i=r+1}^{q} n_i\right] = 0, \quad j = 1, \dots, m^*. \tag{39}$$

Eq. (39) represents the analytical condition to be fulfilled by the generic orientation group G_j^* of the macro-sequence in order to be a saturated one. If all orientation groups satisfy Eq. (39) then the macro-sequence is a QT uncoupled stack.

To the best of the authors knowledge, this is the first time that a completely general analytical condition is determined for obtaining QT solutions derived by the superposition of other QT stacks.

5.1. Special case: superposition of 2 QT sequences ($q = 2$)

When imposing $q = 2$, Eq. (39) reduces to:

$$-n_{G_j^{(1)}} n_2 + n_{G_j^{(2)}} n_1 = 0, \quad j = 1, \dots, m^*, \tag{40}$$

or even:

$$\frac{n_{G_j^{(1)}}}{n_1} = \frac{n_{G_j^{(2)}}}{n_2}, \quad j = 1, \dots, m^*. \tag{41}$$

Eq. (41) represents a general condition to obtain an uncoupled QT macro-sequence by superposing two elementary QT sequences. Some remarks can be drawn:

1. the initial QT sequences should possess exactly the same orientations. If orientation θ_j is present in only one of the two sequences, Eq. (41) is not satisfied and the associated group is not saturated;
2. it must be assured that, for each j-th orientation, in both QT_1 and QT_2 an equal percentage of plies oriented at θ_j is present, regardless the position of those plies. In the particular case of $n_1 = n_2$ Eq. (41) imposes that the two sequences should have the same number of plies for each orientation;
3. no limitations arise on the number of groups that can be involved;
4. the superposition of sequences with $n_1 \neq n_2$ is still possible, allowing for greater design freedom.

To be remarked that the superposition of a single QT sequence repeated two times falls under this case study: the macro-sequence obtained is still a QT uncoupled solution. This is still true if the sequence is reversed and repeated, so as to have a final symmetric macro-sequence.

5.2. Special case: superposition of 3 QT sequences ($q = 3$)

A second case study, for $q = 3$, is presented here. However, firstly, a remark valid for all cases with $q \geqslant 3$ can be done: suppose there is an orientation group which is present in only one of the initial QT sequences, e.g. the r-th. Therefore it is:

$$n_{G_j^{(i)}} = 0, \qquad \forall\; i \neq r. \tag{42}$$

It follows from Eq. (39):

$$n_{G_j^{(r)}}\Big(\sum_{i=1}^{r-1} n_i - \sum_{i=r+1}^{q} n_i\Big) = 0. \tag{43}$$

Clearly the condition in Eq. (43) can be satisfied when:

$$\sum_{i=1}^{r-1} n_i = \sum_{i=r+1}^{q} n_i. \tag{44}$$

This condition means that in the macro-sequence the number of plies below and above the r-th sequence should be equal, that is the r-th elementary QT sequence must be exactly at the center of the macro-sequence.

Going further, for $q = 3$ Eqs. (39) writes:

$$n_1(-n_{G_j^{(2)}} - n_{G_j^{(3)}}) + n_2(n_{G_j^{(1)}} - n_{G_j^{(3)}}) + n_3(n_{G_j^{(1)}} + n_{G_j^{(2)}}) = 0, \quad j = 1, \ldots, m^*. \tag{45}$$

From this relationship some remarks can be easily inferred:

- it is immediate to verify that if $n_{G_j^{(1)}} = n_{G_j^{(3)}} = 0$ and $n_1 = n_3$ then $n_{G_j^{(2)}}$ can be different from zero to satisfy the condition;
- if $n_{G_j^{(1)}} = n_{G_j^{(3)}} \neq 0$, the number of layers of the second elementary QT stack, i.e. n_2, does not affect the nature of the macro-stack;;
- if $n_1 = n_2 = n_3$, the condition simplifies to $n_{G_j^{(1)}} = n_{G_j^{(3)}}$.

6. Homogeneity of superposed QT stacking sequences

6.1. Superposition of homogeneous QT sequences

Following the same idea of Section 5 in this Section an analytical criterion for obtaining macro-stacks with $\mathbf{C} = \mathbf{0}$ by simply superposing QT homogeneous elementary solutions is derived, too. For the generic r-th elementary stack QT_r, Eq. (17) applies as follows:

$$\sum_{k \in G_j^{(r)}} c_k = 0, \qquad \begin{cases} \forall\;\; r = 1, \ldots, q, \\ \forall\;\; j = 1, \ldots, m_r. \end{cases} \tag{46}$$

Taking into account the general expression of c_k given in Eq. (11), Eq. (46) becomes:

$$\sum_{k \in G_j^{(r)}} -2n_r^2 - 12k^2 + 12kn_r + 12k - 4 - 6n_r = 0, \qquad \begin{cases} \forall\;\; r = 1, \ldots, q, \\ \forall\;\; j = 1, \ldots, m_r. \end{cases} \tag{47}$$

In order for the macro-sequence to be a QT homogeneous one, each orientation group should be saturated within the macro-stack frame.. This requirement can be formalised as:

$$\sum_{k^* \in G_j^*} c_{k^*} = 0, \quad j = 1, \ldots, m^*. \tag{48}$$

By utilising the properties of Eq. (26), the sum over G_j^* in Eq. (48) can be split in multiple sum over $G_j^{*,(r)}$:

$$\sum_{k^* \in G_j^*} c_{k^*} = \sum_{r=1}^{q} \sum_{k^* \in G_j^{*,(r)}} c_{k^*} = 0, \quad j = 1, \ldots, m^*. \tag{49}$$

Then considering Eq. (30), the previous relationship writes:

$$\sum_{r=1}^{q} \sum_{k^* \in G_j^{*,(r)}} c_{k^*} = \sum_{r=1}^{q} \sum_{k \in G_j^{(r)}} c_{k+\Delta k^{(r)}} = 0, \quad j = 1, \ldots, m^*. \tag{50}$$

Replacing Eqs. (11) (carefully adapted to the notation of the macro-sequence) and Eq. (24) into Eq. (50), one obtains:

$$\sum_{r=1}^{q} \sum_{k \in G_j^{(r)}} [-2n_{tot}^2 - 12(k + \Delta k^{(r)})(k + \Delta k^{(r)} - n_{tot} - 1) - 4 - 6n_{tot}] = 0, \quad j = 1, \ldots, m_r. \tag{51}$$

Eq. (51) can be further simplified by taking into account Eq. (47):

$$\sum_{r=1}^{q} \sum_{k \in G_j^{(r)}} [2(n_r^2 - n_{tot}^2) + 6(n_r - n_{tot}) - 12k(n_r - n_{tot} + 2\Delta k^{(r)}) - 12\Delta k^{(r)}(\Delta k^{(r)} - n_{tot} - 1)] = 0, \quad j = 1, \ldots, m_r. \tag{52}$$

By introducing $n_{G_j^{(r)}}$, i.e. the number of plies oriented at θ_j in the r-th initial sequence, into Eq. (52) and after some algebraic manipulations Eq. (52) becomes:

$$\sum_{r=1}^{q} \left\{ n_{G_j^{(r)}} \left[12\Delta k^{(r)} \left(n_r + 1 + \sum_{i=r+1}^{q} n_i \right) - 2\left(\Delta k^{(r)} + \sum_{i=r+1}^{q} n_i \right)(n_r + 3 + n_{tot}) \right] - \sum_{k \in G_j^{(r)}} 12k \left(\Delta k^{(r)} - \sum_{i=r+1}^{q} n_i \right) \right\} = 0, \quad j = 1, \ldots, m_r. \tag{53}$$

Generally speaking, the macro-stack obtained by superposing q elementary homogeneous QT solutions is a homogeneous QT sequence if Eq. (53) is met. However, such condition appears to be more complex than the corresponding counterpart derived for a QT uncoupled macro-sequence. In this case the term k, appearing in Eq. (53), introduces the influence of the position of the plies belonging to the considered j-th orientation group.

6.1.1. Special case: superposition of 2 QT sequences ($q = 2$)

When $q = 2$ Eq. (53) reduces to:

$$n_{G_j^{(1)}}[-2n_2(n_1 + n_{tot} + 3)] + \sum_{k \in G_j^{(1)}} 12kn_2 + n_{G_j^{(2)}}[12n_1(n_2 + 1) - 2n_1(n_2 + n_{tot} + 3)] - \sum_{k \in G_j^{(2)}} 12kn_1 = 0, \quad j = 1, \ldots, m^*. \tag{54}$$

In the particular case of $n_1 = n_2$ and $n_{G_j^{(1)}} = n_{G_j^{(2)}}$ it follows:

$$\sum_{k \in G_j^{(1)}} k - \sum_{k \in G_j^{(2)}} k = 0, \quad j = 1, \ldots, m^*. \tag{55}$$

Eq. (55) simply means that, in order to obtain an homogeneous QT macro-sequence, the sums of k indexes must be the same for the two initial sequences.

6.1.2. Special case: superposition of 3 QT sequences ($q = 3$)

In the case of 3 QT homogeneous solutions to be superposed Eq. (53) writes:

$$n_{G_j^{(1)}}[-2(n_2 + n_3)(n_1 + n_{tot} + 3)] + \sum_{k \in G_j^{(1)}} 12k(n_2 + n_3) + n_{G_j^{(2)}}[12n_1(n_2 + n_3 + 1) - 2(n_1 + n_3)(n_2 + n_{tot} + 3)] - \sum_{k \in G_j^{(2)}} 12k(n_1 - n_3) + n_{G_j^{(3)}}[12(n_1 + n_2)(n_3 + 1) - 2(n_1 + n_2)(n_3 + n_{tot} + 3)] - \sum_{k \in G_j^{(3)}} 12k(n_1 + n_2) = 0, \quad j = 1, \ldots, m^*. \tag{56}$$

If $n_1 = n_2 = n_3$ and $n_{G_j^{(1)}} = n_{G_j^{(2)}} = n_{G_j^{(3)}}$, Eq. (56) becomes:

$$\sum_{k \in G_j^{(1)}} k - \sum_{k \in G_j^{(3)}} k = 0, \quad j = 1, \ldots, m^*. \tag{57}$$

In this particular case, the contribution of sequence QT_2 disappears, and only k indexes of sequences QT_1 and QT_3 must fulfil the previous condition in order to get a QT homogeneous macro-stack.

6.2. Superposition of quasi-homogeneous QT sequences

The homogeneity criterion of Eq. (53) can be further simplified for the particular case of superposition of quasi-homogeneous QT elementary stacks. Such a criterion can be obtained by imposing the condition of uncoupling of Eq. (32) to Eq. (53). Indeed, firstly Eq. (53) can be rewritten as:

$$\sum_{r=1}^{q} \left\{ n_{G_j^{(r)}} \left[12\Delta k^{(r)} \left(n_r + 1 + \sum_{i=r+1}^{q} n_i \right) - 2\left(\Delta k^{(r)} + \sum_{i=r+1}^{q} n_i \right) (n_r + 3 + n_{tot}) \right] - \sum_{k \in G_j^{(r)}} (12k + 6n_r + 6 - 6n_r - 6) \left(\Delta k^{(r)} - \sum_{i=r+1}^{q} n_i \right) \right\} = 0, \quad j = 1, \ldots, m_r, \tag{58}$$

and injecting Eq. (32) it follows:

$$\sum_{r=1}^{q} \left\{ n_{G_j^{(r)}} \left[\left(\Delta k^{(r)} + \sum_{i=r+1}^{q} n_i \right) \left(\Delta k^{(r)} + \sum_{i=r+1}^{q} n_i - n_r \right) - 6\Delta k^{(r)} \sum_{i=r+1}^{q} n_i \right] \right\} = 0, \quad j = 1, \ldots, m_r. \tag{59}$$

Of course the criterion expressed in Eq. (59) is simpler than the one of Eq. (53) thanks to the integration of uncoupling condition. It is noteworthy that, the quasi-homogeneity of QT elementary solutions does not imply the quasi-homogeneity of the resulting macro-sequence.

6.2.1. Special case: superposition of 2 QT sequences ($q = 2$)

When $q = 2$ Eq. (59) simplifies to:

$$(n_1 - n_2)(n_1 n_{G_j^{(2)}} - n_2 n_{G_j^{(1)}}) = 0, \quad j = 1, \ldots, m^*. \tag{60}$$

Clearly this condition can be split into:

$$\begin{cases} (n_1 - n_2) = 0, \\ (n_1 n_{G_j^{(2)}} - n_2 n_{G_j^{(1)}}) = 0, \quad j = 1, \ldots, m^*. \end{cases} \tag{61}$$

It is sufficient that only one of the two conditions of Eq. (61) is verified to obtain a QT homogeneous stack. First condition simply requires an equal number of plies for both QT quasi-homogeneous solutions to be superposed, while the second condition is the same as that expressed by Eq. (41). Therefore, two different situations may arise.

1. The first condition of Eq. (61) is satisfied, but not the second one: the resulting macro-stack is characterised by saturated groups in terms of c_k coefficients, but not for b_k ones. Thus the macro-sequence will be homogeneous but not uncoupled.
2. The second condition of Eq. (61) is satisfied: in this case the macro-sequence satisfy both Eqs. (60) and (41) and thus it is a QT quasi-homogeneous stack.

6.2.2. Special case: superposition of 3 QT sequences ($q = 3$)

When $q = 3$ Eq. (59) reduces to:

$$n_{G_j^{(1)}} [(n_2 + n_3)(n_2 + n_3 - n_1)] + n_{G_j^{(2)}} [(n_1 + n_3)(n_1 + n_3 - n_2) - 6n_1 n_3] + n_{G_j^{(3)}} [(n_1 + n_2)(n_1 + n_2 - n_3)] = 0, j = 1, \ldots, m^*. \tag{62}$$

Let's consider some particular cases. For example, if $n_1 = n_2 = n_3 = n$ Eq. (62) simplifies to:

$$n_{G_j^{(1)}} + n_{G_j^{(3)}} = 2n_{G_j^{(2)}}, \quad j = 1, \ldots, m^*, \tag{63}$$

which is a very simple condition: the sum of plies belonging to a given orientation θ_j in sequences QT_1 and QT_3 must be equal to twice the number of plies sharing the same orientation angle within sequence QT_2. Eq. (63) imposes that a given orientation group G_j must be present in the central sequence too, otherwise saturation will not be possible. This is due to the trend of c_k coefficients within the stacking sequence.

7. Quasi-homogeneity of superposed QT stacking sequences

The macro-stack obtained by superposition of elementary QT solutions is quasi-homogeneous if conditions of Eqs. (33) and (48) are simultaneously met, namely:

$$\begin{cases} \sum_{k^* \in G_j^*} b_{k^*} = 0, \\ \sum_{k^* \in G_j^*} c_{k^*} = 0, \end{cases} j = 1, \ldots, m^*. \tag{64}$$

These two requirements, applied to the special case of superposition of quasi-homogeneous QT elementary stacks, give rise to a set of two equations to be satisfied, i.e. Eq. (39) for uncoupling and Eq. (59) for homogeneity.

If Eqs. (39) and (59) are both satisfied the generic orientation group θ_j (within the resulting macro-sequence) becomes a saturated group in terms of both coefficients b_k and c_k.

8. Numerical examples

In this section some numerical examples exploiting the results obtained in the previous sections are given. In all examples, a unidirectional carbon-epoxy ply is considered whose properties are listed in Table 5.

To perform calculations the following orientations are associated to each saturated group:

$1 \rightarrow 10°$
$2 \rightarrow 35°$
$3 \rightarrow -65°$

Of course this choice of orientations is absolutely arbitrary, indeed if different angles are chosen the properties of uncoupling, homogeneity or quasi-homogeneity are always verified (since they have been obtained regardless to the value of the ply orientation angle).

8.1. Superposition of 2 uncoupled QT sequences ($q = 2$)

For this first example, the following two QT uncoupled sequences will be taken into account:

$QT_1 = [1\ 2\ 2\ 1\ 3\ 3\ 2\ 1\ 1\ 2]$,
$n_1 = 10, \quad n_{G_1^{(1)}} = 4, \quad n_{G_2^{(1)}} = 4, \quad n_{G_3^{(1)}} = 2,$

$QT_2 = [1\ 2\ 3\ 2\ 1\ 2\ 1\ 3\ 1\ 2]$,
$n_2 = 10, \quad n_{G_1^{(2)}} = 4, \quad n_{G_2^{(2)}} = 4, \quad n_{G_3^{(2)}} = 2.$

Superposing the two sequences, according to the rule of Eq. (39) one

Table 5
Constitutive lamina properties.

E_{11} [Gpa]	E_{22} [Gpa]	G_{12} [Gpa]	ν_{12}
138.0	8.97	6.9	0.30
$t_{ply} = 0.127$ mm			

obtains:

[1 2 2 1 3 3 2 1 1 2 1 2 3 2 1 2 1 3 1 2],

with:

$n_{tot} = 20, \quad n_{G_1^*} = 8, \quad n_{G_2^*} = 8, \quad n_{G_3^*} = 4,$

and:

$\mathbf{B}^* = \mathbf{0} \quad [GPa],$

and thus this sequence is really uncoupled, as expected according to Eq. (41).

Consider now a more general situation, characterised by two QT uncoupled sequences with a different number of plies:

$QT_1 = [1\ 2\ 2\ 1\ 3\ 3\ 2\ 1\ 1\ 2],$
$n_1 = 10, \quad n_{G_1^{(1)}} = 4, \quad n_{G_2^{(1)}} = 4, \quad n_{G_3^{(1)}} = 2,$

$QT_2 = [1\ 2\ 1\ 2\ 3\ 3\ 2\ 1\ 2\ 1\ 2\ 1\ 3\ 1\ 2],$
$n_2 = 15, \quad n_{G_1^{(2)}} = 6, \quad n_{G_2^{(2)}} = 6, \quad n_{G_3^{(2)}} = 3.$

By superposing the two sequences the following one is obtained:

[1 2 2 1 3 3 2 1 1 2 1 2 1 2 3 3 2 1 2 1 2 1 3 1 2],
$n_{tot} = 25, \quad n_{G_1^*} = 10, \quad n_{G_2^*} = 10, \quad n_{G_3^*} = 5,$

with:

$\mathbf{B}^* = \mathbf{0} \quad [GPa].$

This last example clearly shows how simple is the utilisation of criterion (41) to obtain a QT sequence even in the case of superposition of uncoupled QT solutions with different number of plies.

8.2. Superposition of 3 uncoupled QT sequences ($q = 3$)

In this case the following QT uncoupled elementary solutions are considered:

$QT_1 = [1\ 2\ 2\ 1\ 1\ 1\ 2\ 1\ 1\ 2],$
$n_1 = 10, \quad n_{G_1^{(1)}} = 6, \quad n_{G_2^{(1)}} = 4,$

$QT_2 = [1\ 2\ 1\ 2\ 3\ 3\ 2\ 1\ 2\ 1\ 2\ 1\ 3\ 1\ 2],$
$n_2 = 15, \quad n_{G_1^{(2)}} = 6, \quad n_{G_2^{(2)}} = 6, \quad n_{G_3^{(2)}} = 3,$

$QT_3 = [1\ 2\ 1\ 2\ 1\ 2\ 1\ 1\ 1\ 2],$
$n_3 = 10, \quad n_{G_1^{(3)}} = 6, \quad n_{G_2^{(3)}} = 4.$

When superposing these sequences, for orientation groups θ_1 and θ_2 one can observe that:

$n_{G_j^{(1)}} = n_{G_j^{(3)}} \quad j = 1, 2,$

which, together with $n_1 = n_3$ makes Eq. (45) satisfied for these orientations. Finally Eq. (45) is satisfied also for orientation θ_3, because it is present only in sequence QT_2. As expected, for the superposition of the three sequences it results $\mathbf{B}^* = \mathbf{0}$.

8.3. Superposition of 2 homogeneous QT sequences ($q = 2$)

As an example for this case the following sequences can be used:

$QT_1 = [1\ 2\ 1\ 1\ 2\ 1\ 2\ 2\ 1\ 2],$
$n_1 = 10, \quad n_{G_1^{(1)}} = 5, \quad n_{G_2^{(1)}} = 5,$

$QT_2 = [1\ 1\ 2\ 2\ 1\ 2\ 1\ 1\ 2\ 2],$
$n_2 = 10, \quad n_{G_1^{(2)}} = 5, \quad n_{G_2^{(2)}} = 5.$

These sequences are both composed by 10 plies and 2 orientation groups. The sums of k indexes of the plies belonging to each orientation group is such that Eq. (55) is satisfied. The resulting macro-sequence is then homogeneous but not uncoupled, in fact the normalised laminate stiffness matrices are:

$$\mathbf{A}^* = \mathbf{D}^* = \begin{bmatrix} 101,15 & 17,06 & 30,02 \\ 17,06 & 17,98 & 11,57 \\ 30,02 & 11,57 & 21,25 \end{bmatrix},$$

$$\mathbf{B}^* = \begin{bmatrix} -2,73 & 0,990 & 0,873 \\ 0,990 & 0,756 & 0,873 \\ 0,873 & 0,873 & 0,990 \end{bmatrix} \quad [GPa]$$

8.4. Superposition of 3 homogeneous QT sequences ($q = 3$)

For this case, the following QT homogeneous elementary stacks are superposed:

$QT_1 = [1\ 2\ 3\ 2\ 1\ 1\ 3\ 3\ 3\ 2\ 1\ 1\ 1\ 3\ 3],$
$n_1 = 15, \quad n_{G_1^{(1)}} = 6, \quad n_{G_2^{(1)}} = 3, \quad n_{G_3^{(1)}} = 6,$

$QT_2 = [1\ 2\ 3\ 2\ 1\ 2\ 3\ 3\ 3\ 1\ 1\ 1\ 1\ 3\ 3],$
$n_2 = 15, \quad n_{G_1^{(2)}} = 6, \quad n_{G_2^{(2)}} = 3, \quad n_{G_3^{(2)}} = 6,$

$QT_3 = [1\ 2\ 3\ 2\ 3\ 1\ 1\ 1\ 3\ 2\ 1\ 3\ 3\ 3\ 1],$
$n_3 = 15, \quad n_{G_1^{(3)}} = 6, \quad n_{G_2^{(3)}} = 3, \quad n_{G_3^{(3)}} = 6.$

It is easy to verify that the sums of k indexes of plies of each orientation group are equal for sequences QT_1 and QT_3. Then, Eq. (57) is satisfied. For the macro-sequence, the following result is obtained:

$$\mathbf{A}^* = \mathbf{D}^* = \begin{bmatrix} 72,91 & 15,86 & 11,78 \\ 15,86 & 48,62 & -10,59 \\ 11,78 & -10,59 & 20,05 \end{bmatrix} \quad [GPa]$$

as expected.

8.5. Superposition of 2 quasi-homogeneous QT sequences ($q = 2$)

The following QT quasi-homogeneous elementary solutions are considered:

$QT_1 = [1\ 2\ 1\ 1\ 1\ 2\ 1],$
$n_1 = 7, \quad n_{G_1^{(1)}} = 5, \quad n_{G_2^{(1)}} = 2,$

$QT_2 = [1\ 2\ 1\ 1\ 1\ 1\ 1\ 2\ 2\ 2\ 1\ 1\ 1\ 1\ 1\ 2\ 1\ 1\ 1\ 1\ 2],$
$n_2 = 21, \quad n_{G_1^{(2)}} = 15, \quad n_{G_2^{(2)}} = 6.$

The first one is the only quasi-homogeneous independent solution existing in the case of a laminate composed of seven plies, while the second is a 21 plies solution with three orientations group, which have been reduced to two by assuming θ_3 equal to orientation θ_1. These two sequences satisfy the second condition in Eq. (61). Indeed, when superposing them, the following results are obtained:

$$\mathbf{A}^* = \mathbf{D}^* = \begin{bmatrix} 114,2 & 12,35 & 25,86 \\ 12,35 & 14,38 & 7,41 \\ 25,86 & 7,41 & 16,54 \end{bmatrix}, \mathbf{B}^* = \mathbf{0}, \quad [GPa].$$

Another interesting example for this case may be the following one. Consider two solutions for the case of 13 plies and 3 orientation groups:

$QT_1 = [1\ 2\ 3\ 1\ 1\ 3\ 1\ 2\ 1\ 1\ 2\ 3\ 1],$
$n_1 = 13, \quad n_{G_1^{(1)}} = 7, \quad n_{G_2^{(1)}} = 3, \quad n_{G_3^{(1)}} = 3,$

and

$QT_2 = [1\ 2\ 3\ 1\ 3\ 1\ 3\ 2\ 1\ 1\ 2\ 1\ 3],$
$n_2 = 13, \quad n_{G_1^{(2)}} = 6, \quad n_{G_2^{(2)}} = 3, \quad n_{G_3^{(2)}} = 4.$

For these sequences the first condition in (61) is satisfied, while the second one is not. Therefore the superposition of this sequences is expected to be homogeneous but not uncoupled. Indeed, the laminate stiffness matrices for the macro-stack are:

$$\mathbf{A}^* = \mathbf{D}^* = \begin{bmatrix} 86,23 & 14,77 & 16,44 \\ 14,77 & 37,47 & -4,65 \\ 16,44 & -4,65 & 18,96 \end{bmatrix},$$
$$\mathbf{B}^* = \begin{bmatrix} -2,23 & 0,259 & -0,597 \\ 0,259 & 1,72 & -0,786 \\ -0,786 & 0,786 & 0,259 \end{bmatrix} \quad [GPa].$$

8.6. Superposition of 3 quasi-homogeneous QT sequences ($q = 3$)

For this case, consider the three QT quasi-homogeneous solutions with 23 layers and 2 saturated groups reported here below:

$QT_1 = [1\ 1\ 1\ 2\ 2\ 2\ 1\ 2\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 1\ 2\ 2\ 1\ 1\ 1\ 2]$,
$n_1 = 23, \quad n_{G_1^{(1)}} = 14, \quad n_{G_2^{(1)}} = 9,$

$QT_2 = [1\ 1\ 1\ 2\ 2\ 2\ 2\ 2\ 1\ 1\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 2\ 2\ 1\ 2\ 1]$,
$n_2 = 23, \quad n_{G_1^{(2)}} = 13, \quad n_{G_2^{(2)}} = 10,$

$QT_3 = [1\ 2\ 1\ 1\ 2\ 2\ 2\ 2\ 1\ 2\ 1\ 1\ 1\ 1\ 1\ 2\ 2\ 1\ 2\ 2\ 1\ 2\ 1]$,
$n_3 = 23, \quad n_{G_1^{(3)}} = 12, \quad n_{G_2^{(3)}} = 11.$

It is evident that both the orientation groups satisfy Eq. (63). When superposing the three sequences, with stack QT_2 at the center, one obtains:

$$\mathbf{A}^* = \mathbf{D}^* = \begin{bmatrix} 105,1 & 15,62 & 28,76 \\ 15,62 & 16,88 & 10,30 \\ 28,76 & 10,30 & 19,82 \end{bmatrix},$$
$$\mathbf{B}^* = \begin{bmatrix} -1,17 & 0,425 & 0,375 \\ 0,425 & 0,325 & 0,375 \\ 0,375 & 0,375 & 0,425 \end{bmatrix} \quad [GPa].$$

The normalised membrane/bending coupling matrix is not null because Eq. (45) is not satisfied.

A simple example of three sequences satisfying both Eqs. (45) and (63) is the following one:

$QT_1 = [1\ 1\ 1\ 2\ 2\ 2\ 1\ 2\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 1\ 2\ 2\ 1\ 1\ 1\ 2]$,
$n_1 = 23, \quad n_{G_1^{(1)}} = 14, \quad n_{G_2^{(1)}} = 9,$

$QT_2 = [1\ 2\ 1\ 1\ 2\ 2\ 1\ 1\ 2\ 1\ 2\ 1\ 1\ 1\ 1\ 2\ 1\ 2\ 2\ 1\ 1\ 2\ 1]$,
$n_2 = 23, \quad n_{G_1^{(2)}} = 14, \quad n_{G_2^{(2)}} = 9,$

$QT_3 = [1\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 2\ 1\ 1\ 2\ 1\ 1\ 1\ 2\ 1\ 1\ 1\ 2\ 1\ 2]$,
$n_3 = 23, \quad n_{G_1^{(3)}} = 14, \quad n_{G_2^{(3)}} = 9.$

In this case both Eqs. (45) and (63) are satisfied and the macro-sequence is still a QT quasi-homogeneous one, for which:

$$\mathbf{A}^* = \mathbf{D}^* = \begin{bmatrix} 107,8 & 14,67 & 27,91 \\ 14,67 & 16,15 & 9,46 \\ 27,91 & 9,46 & 18,86 \end{bmatrix}, \mathbf{B}^* = \mathbf{0} \quad [GPa].$$

9. Conclusions

In this paper, many improvements in the field of QT solutions have been proposed. These advances are interesting and particularly useful when dealing with the design problem of thick laminates.

Firstly, an improved version of the algorithm, initially presented in [1], has been implemented. The algorithm is able to find a higher number of QT independent solutions (for a given combination of both plies and orientation groups number) when compared to the algorithm proposed in [1]. On the other hand it is possible to extend the database of QT solutions (for each considered case) up to a higher number of layers than in the past. In this way a significant database of QT stacking sequences can be constituted.

Secondly, exact analytical rules to generate QT sequences by superposition of QT elementary stacks have been derived for the cases of uncoupling, homogeneity and quasi-homogeneity. These rules are very easy to be applied and constitute a major improvement for the utilisation of QT sequences in the design of thick laminates. In fact, they allow designing QT sequences with an arbitrarily (high) number of plies, without the need of using the algorithm for generating QT solutions. Indeed, for a number of layers higher than 35 this task is quite complicated due to excessively high computational resources required.

Furthermore, thanks to their simplicity and general applicability, these rules can be used in early stages of design. It is noteworthy that QT solutions found by using the superposition criteria presented in this study do not constitute the overall number of QT solutions for a given number of plies.

Finally, to prove the effectiveness and the exactness of the derived rules, some meaningful numerical examples have been proposed. These examples aim also at giving a deeper insight into the matter and constitute a sort of “guidelines” in the proper application of the analytical rules for superposing QT solutions.

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