The Hankel transform of first- and second-order tensor fields: definition and use for modeling circularly symmetric leaky waveguides

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Abstract

A *tensor Hankel transform* (THT) is defined for vector fields, such as displacement, and second-order tensor fields, such as stress or strain. The THT establishes a bijection between the real space and the wave-vector domain, and, remarkably, cannot be reduced to a scalar transform applied separately to each component.

One of the advantages of this approach is that some standard elasticity problems can be concisely rewritten by applying this tensor integral transform coupled with an azimuthal Fourier series expansion. A simple and compact formulation of the boundary conditions is also achieved.

Thanks to the THT, we obtain for each azimuthal wavenumber and each azimuthal direction exactly the same wave equation as for a standard 2D model of elastic wave propagation. Thus, waves similar to the standard plane P, SV and SH waves are naturally found.

Lastly, the THT is used to calculate the ultrasonic field in an isotropic cylindrical leaky waveguide, the walls of which radiating into a surrounding elastic medium, by using a standard scattering approach.

Keywords

Circular symmetric waveguide, Cylindrical coordinates, Elastic Wave Scattering, P, SV and SH waves, Tensor Hankel transform, 2D Fourier transform.

1 Introduction

The modeling of elastic wave propagation in circularly symmetric media can be done using space integral transform in cylindrical coordinates and does not necessitate a potential formulation. In this context, The Hankel transform of a scalar field has been well known for a long time. It appears when the 2D Fourier transform of a scalar field is rewritten in polar coordinates (*e.g.*, [1], [2]). Similarly, a *tensor Hankel transform* (THT) can be naturally defined for vector fields and matrix fields. Even if we believe that such a tensor integral transform was probably published between 1920 and 1960, we did not find in the literature neither the elaboration of the concept of THT nor the Hankel transform of a second-order tensor. In our knowledge, the Hankel transform of a vector field was introduced in electromagnetism only thirty years ago ([3], followed by [4], [5]). We only found it in a recent book [6] in an implicit formulation for elasticity applications. In the first section, definitions and properties of the THT are given for scalar, vector, and matrix fields.

In the second section, the THT is applied to elastodynamics. By using it, some standard elasticity problems can be concisely rewritten. A direct link is notably established between a cylinder in the radial-wavenumber(k)/ axial-position(z)/frequency(ω) domain and the standard 2D problem. Thus, the plane P, SV and SH waves can be transposed for a circularly symmetric geometry, for each azimuthal wavenumber and each azimuthal direction.

The THT is also convenient for easily achieving the ultrasonic field generated in an elastic half-space by a transducer in contact with it. A direct application is the modeling of propagation in an isotropic cylindrical leaky waveguide, the walls of which radiating into a surrounding elastic medium. Indeed, the incident wave generated at the end of the cylinder is firstly calculated in the (k, z, ω)–domain and then rewritten in the (r, k_z, ω)–domain, k_z denoting the axial wavenumber. Lastly, the reflected (guided) wave in the cylinder and the transmitted wave radiated into the surrounding media are deduced. Only principles are described here. Detailed calculation, approximations and numerical results are given in a related paper [7].

2 Definitions

2.1 Hankel transform of a scalar field

The standard 2D Fourier transform of a scalar field ϕ is an integral transform which establishes a correspondence between ϕ in the real space and $\hat{\phi}$ in the wave-vector domain, both in Cartesian coordinates (see Fig.1):

$$\underbrace{\hat{\phi}(\mathbf{k}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \phi(\mathbf{x})\, \mathrm{e}^{\,\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}\, \mathrm{d}\mathbf{x}} \\ \Updownarrow \\ \overbrace{\phi(\mathbf{x}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \hat{\phi}(\mathbf{k})\, \mathrm{e}^{-\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}\, \mathrm{d}\mathbf{k}} \quad , \tag{1}$$

where $\mathbf{x}$, $\mathbf{k}$ and the dot denote the position $\begin{pmatrix} x \\ y \end{pmatrix}$, the wave-vector $\begin{pmatrix} k_x \\ k_y \end{pmatrix}$, and the standard scalar product, respectively.

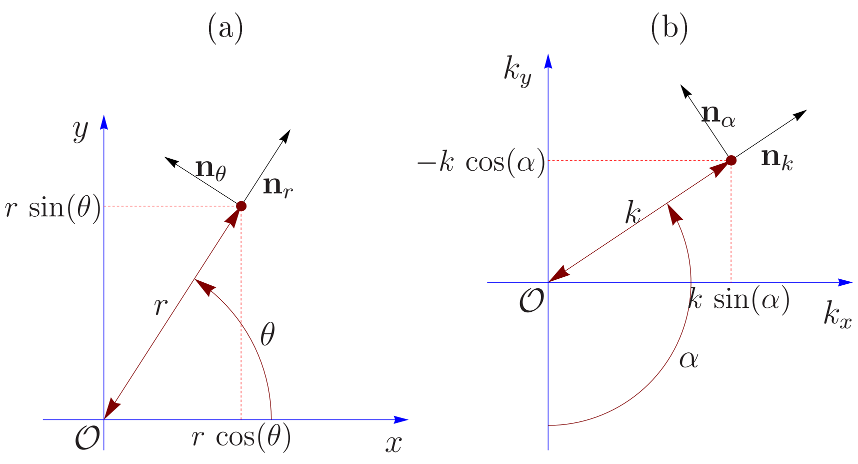

Figure 1: Cartesian and polar coordinates (a) in the real space and (b) in the wave-vector domain.

In polar coordinates (see Fig.1), the azimuthal Fourier series expansion of the scalar field ϕ leads to:

$$\phi(\mathbf{x}) = \phi_0(r) + \sum_{n=1}^{+\infty} \cos(n\,\theta)\;\phi_n^{\vdash}(r) + \sin(n\,\theta)\;\phi_n^{\perp}(r)\ , \tag{2.a}$$

with

$$\phi_0(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(\mathbf{x})\,\mathrm{d}\theta\,, \tag{2.b}$$

$$\phi_n^{\vdash}(r) = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(n\,\theta)\;\phi(\mathbf{x})\,\mathrm{d}\theta\,, \tag{2.c}$$

$$\phi_n^{\perp}(r) = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(n\,\theta)\;\phi(\mathbf{x})\,\mathrm{d}\theta\,. \tag{2.d}$$

The 2D Fourier transform (1) becomes:

$$\hat{\phi}(\mathbf{k}) = \int_0^{+\infty} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(\mathbf{x})\, \mathbb{e}^{\,\mathbb{i}\, k\, r\, \sin(\alpha-\theta)}\, \mathrm{d}\theta \right] r\, \mathrm{d}r \tag{3}$$

and, by using the integral representation (*e.g.*, [8, §17.23]) of the Bessel function of the first kind:

$$\mathcal{J}_n(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbb{e}^{\,\mathbb{i}\,(\, a\, \sin(\beta) - n\, \beta\,)}\, \mathrm{d}\beta \tag{4}$$

and the property $\mathcal{J}_{-n}(a) = (-1)^n \mathcal{J}_n(a)$, we obtain after some algebra the azimuthal Fourier series expansion of $\hat{\phi}$:

$$\begin{aligned} \hat{\phi}(\mathbf{k}) \;=\; & \Phi_0(k) + \sum_{\substack{n=2 \\ \text{even}\, n}}^{+\infty} \cos(n\,\alpha)\ \Phi_n^{\vdash}(k) + \sin(n\,\alpha)\ \Phi_n^{\perp}(k) \\ & + \mathbb{i} \sum_{\substack{n=1 \\ \text{odd}\, n}}^{+\infty} - \cos(n\,\alpha)\ \Phi_n^{\perp}(k) + \sin(n\,\alpha)\ \Phi_n^{\vdash}(k)\ , \end{aligned} \tag{5}$$

the coefficients of which being the n^{th}-order Hankel transform (*e.g.*, [1], [2]) of the coefficients in (2):

$$\begin{gathered} \underbrace{\Phi_n^{\perp,\vdash}(k) = \int_0^{+\infty} \phi_n^{\perp,\vdash}(r)\, \mathcal{J}_n(k\, r)\, r\, \mathrm{d}r} \\ \Updownarrow \\ \overbrace{\phi_n^{\perp,\vdash}(r) = \int_0^{+\infty} \Phi_n^{\perp,\vdash}(k)\, \mathcal{J}_n(k\, r)\, k\, \mathrm{d}k}\quad . \end{gathered} \tag{6}$$

Note the dependence on the parity of n in (5) because the azimuth of the wave-vector $-\mathbf{k}$ is $\alpha+\pi$.

2.2 THT of a first-order tensor field

A vector field $\mathbf{u}$ and its 2D Fourier transform $\hat{\mathbf{u}}$ are written in cylindrical coordinates:

$$\mathbf{u}(\mathbf{x}) = \begin{bmatrix} u_r(\mathbf{x}) \\ u_\theta(\mathbf{x}) \\ u_z(\mathbf{x}) \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{u}}(\mathbf{k}) = \begin{bmatrix} \mathbb{i}\, U_k(\mathbf{k}) \\ \mathbb{i}\, U_\alpha(\mathbf{k}) \\ U_z(\mathbf{k}) \end{bmatrix} = \begin{pmatrix} \mathbb{i} \\ \mathbb{i} \\ 1 \end{pmatrix} \star \mathbf{U}(\mathbf{k})\ , \tag{7}$$

the change-of-basis matrices being (see Fig.1):

$$\mathcal{R} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \mathcal{S} = \begin{pmatrix} \sin\alpha & \cos\alpha & 0 \\ -\cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\ , \tag{8}$$

and $\star$ denoting the elementwise product.

Coefficients $\mathbb{i}$ in (7) ensure that each component U_ξ of $\mathbf{U}$ satisfies $U_\xi(-\mathbf{k}) = U_\xi^*(\mathbf{k})$ (complex conjugate) if the vector field $\mathbf{u}$ is real-valued.

The 2D transform of the vector field $\mathbf{u}$ is:

$$\hat{\mathbf{u}}(\mathbf{k}) = \int_0^{+\infty} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbb{e}^{\,\mathbb{i}\, k\, r\, \sin(\alpha-\theta)}\, \mathcal{S}^T \mathcal{R}\, \mathbf{u}(\mathbf{x})\, \mathrm{d}\theta \right] r\, \mathrm{d}r\ , \tag{9}$$

Its azimuthal Fourier series expansion gives:

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}_0(r) + \sum_{n=1}^{+\infty} \mathbf{Cos}(n\theta) \star \mathbf{u}_n^{\vdash}(r) + \mathbf{Sin}(n\theta) \star \mathbf{u}_n^{\perp}(r)\ , \tag{10}$$

where $\mathbf{Cos}(a)=\begin{pmatrix}\cos a\\ \sin a\\ \cos a\end{pmatrix}$, $\mathbf{Sin}(a)=\begin{pmatrix}\sin a\\ -\cos a\\ \sin a\end{pmatrix}$.

After some algebra, (4), (9) and (10) yield the azimuthal Fourier series expansion of $\mathbf{U}(\mathbf{k})$:

$$\begin{aligned}\mathbf{U}(\mathbf{k}) = \mathbf{U}_0(k) + \sum_{\substack{n=2\\ \text{even}\, n}}^{+\infty} \mathbf{Cos}(n\alpha)\star\mathbf{U}_n^{\vdash}(k) + \mathbf{Sin}(n\alpha)\star\mathbf{U}_n^{\perp}(k)\\ +\,\mathbb{i}\sum_{\substack{n=1\\ \text{odd}\, n}}^{+\infty} -\mathbf{Cos}(n\alpha)\star\mathbf{U}_n^{\perp}(k) + \mathbf{Sin}(n\alpha)\star\mathbf{U}_n^{\vdash}(k)\,,\end{aligned} \tag{11}$$

Each vector $\mathbf{U}_n^{\perp,\vdash}$ is the n^{th}-order *vector Hankel transform* [3] [4] [5] of the vector $\mathbf{u}_n^{\perp,\vdash}$:

$$\underbrace{\mathbf{U}_n^{\perp,\vdash}(k) = \int_0^{+\infty} \mathbb{J}_n(k\,r)\,\mathbf{u}_n^{\perp,\vdash}(r)\;r\,\mathrm{d}r}\\ \Updownarrow \\ \overbrace{\mathbf{u}_n^{\perp,\vdash}(r) = \int_0^{+\infty} \mathbb{J}_n(k\,r)\,\mathbf{U}_n^{\perp,\vdash}(k)\;k\,\mathrm{d}k}\;, \tag{12}$$

where the matrix $\mathbb{J}_n$ is defined by:

$$\mathbb{J}_n(a)=\begin{bmatrix}\frac{\mathcal{J}_{n+1}(a)-\mathcal{J}_{n-1}(a)}{2} & \frac{\mathcal{J}_{n+1}(a)+\mathcal{J}_{n-1}(a)}{2} & 0\\ \frac{\mathcal{J}_{n+1}(a)+\mathcal{J}_{n-1}(a)}{2} & \frac{\mathcal{J}_{n+1}(a)-\mathcal{J}_{n-1}(a)}{2} & 0\\ 0 & 0 & \mathcal{J}_n(a)\end{bmatrix}. \tag{13}$$

Note that the THT of a real-valued field is real-valued and that the *Plancherel-Parseval identity* [3] becomes:

$$\int_0^{+\infty} \mathbf{u}_n(r)\cdot\mathbf{v}_n(r)\;r\,\mathrm{d}r = \int_0^{+\infty} \mathbf{U}_n(k)\cdot\mathbf{V}_n(k)\;k\,\mathrm{d}k\;. \tag{14}$$

2.3 THT of a second-order tensor field

A matrix field m and its 2D Fourier transform $\hat{m}$ are written in cylindrical coordinates:

$$m(\mathbf{x}) = \begin{bmatrix} m_{rr}(\mathbf{x}) & m_{r\theta}(\mathbf{x}) & m_{rz}(\mathbf{x})\\ m_{\theta r}(\mathbf{x}) & m_{\theta\theta}(\mathbf{x}) & m_{\theta z}(\mathbf{x})\\ m_{zr}(\mathbf{x}) & m_{z\theta}(\mathbf{x}) & m_{zz}(\mathbf{x})\end{bmatrix} \tag{15}$$

and

$$\hat{m}(\mathbf{k})=\begin{bmatrix} M_{kk}(\mathbf{k}) & M_{k\alpha}(\mathbf{k}) & \mathbb{i}\,M_{kz}(\mathbf{k})\\ M_{\alpha k}(\mathbf{k}) & M_{\alpha\alpha}(\mathbf{k}) & \mathbb{i}\,M_{\alpha z}(\mathbf{k})\\ \mathbb{i}\,M_{zk}(\mathbf{k}) & \mathbb{i}\,M_{z\alpha}(\mathbf{k}) & M_{zz}(\mathbf{k})\end{bmatrix} = \begin{pmatrix}1 & 1 & \mathbb{i}\\ 1 & 1 & \mathbb{i}\\ \mathbb{i} & \mathbb{i} & 1\end{pmatrix}\star\mathcal{M}(\mathbf{k})\,, \tag{16}$$

such that $M_{\xi\zeta}(-\mathbf{k}) = M_{\xi\zeta}^*(\mathbf{k})$ if m is real-valued.

The 2D transform of the matrix field m is:

$$\hat{m}(\mathbf{k})=\int_0^{+\infty}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbb{e}^{\,\mathbb{i}\,k\,r\,\sin(\alpha-\theta)}\;\mathcal{S}^T\mathcal{R}\,m(\mathbf{x})\,\mathcal{R}^T\mathcal{S}\;\mathrm{d}\theta\right]\,r\,\mathrm{d}r\;. \tag{17}$$

Its azimuthal Fourier series expansion yields:

$$m(\mathbf{x})=m_0(r) + \sum_{n=1}^{+\infty} \mathcal{C}os(n\theta)\star m_n^{\vdash}(r) + \mathcal{S}in(n\theta)\star m_n^{\perp}(r)\,, \tag{18.a}$$

where

$$\mathcal{C}os(a)=\begin{pmatrix}\cos a & \sin a & \cos a\\ \sin a & \cos a & \sin a\\ \cos a & \sin a & \cos a\end{pmatrix} \tag{18.b}$$

and

$$\mathcal{S}in(a)=\begin{pmatrix}\sin a & -\cos a & \sin a\\ -\cos a & \sin a & -\cos a\\ \sin a & -\cos a & \sin a\end{pmatrix}. \tag{18.c}$$

With the same approach as above, the *matrix Hankel transform* can be defined as follows:

$$\underbrace{\mathcal{M}_n^{\perp,\vdash}(k) = \int_0^{+\infty} \mathfrak{I}_n(k\,r) : m_n^{\perp,\vdash}(r)\; r\, \mathrm{d}r}_{} \quad \Updownarrow \quad \overbrace{m_n^{\perp,\vdash}(r) = \int_0^{+\infty} \mathfrak{I}_n(k\,r) : \mathcal{M}_n^{\perp,\vdash}(k)\; k\, \mathrm{d}k}^{}\quad , \tag{19}$$

where the colon denotes the tensor product such that $(\mathfrak{T}:\mathcal{P})_{ij} = \sum_{k,l} \mathfrak{T}_{ijkl}\,\mathcal{P}_{kl}$, $\mathfrak{T}$ and $\mathcal{P}$ being fourth-order and second-order tensors, respectively. The fourth-order symmetric tensor $\mathfrak{I}_n$ is defined by ("(a)" is omitted in $\mathcal{J}_\zeta(a)$):

$$\mathfrak{I}_{n,\cdot\cdot 11}(a)=\begin{bmatrix}\frac{-\mathcal{J}_{n-2}+2\mathcal{J}_n-\mathcal{J}_{n+2}}{4} & \frac{\mathcal{J}_{n-2}-\mathcal{J}_{n+2}}{4} & 0\\ \frac{\mathcal{J}_{n-2}-\mathcal{J}_{n+2}}{4} & \frac{\mathcal{J}_{n-2}+2\mathcal{J}_n+\mathcal{J}_{n+2}}{4} & 0\\ 0 & 0 & 0\end{bmatrix}, \tag{20.a}$$

$$\mathfrak{I}_{n,\cdot\cdot 22}(a)=\begin{bmatrix}\frac{\mathcal{J}_{n-2}+2\mathcal{J}_n+\mathcal{J}_{n+2}}{4} & \frac{-\mathcal{J}_{n-2}+\mathcal{J}_{n+2}}{4} & 0\\ \frac{-\mathcal{J}_{n-2}+\mathcal{J}_{n+2}}{4} & \frac{-\mathcal{J}_{n-2}+2\mathcal{J}_n-\mathcal{J}_{n+2}}{4} & 0\\ 0 & 0 & 0\end{bmatrix}, \tag{20.b}$$

$$\mathfrak{I}_{n,\cdot\cdot 33}(a)=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \mathcal{J}_n\end{bmatrix}, \tag{20.c}$$

$$\mathfrak{I}_{n,\cdot\cdot 23}(a)=\mathfrak{I}^T_{n,\cdot\cdot 32}(a)=\begin{bmatrix}0 & 0 & \frac{\mathcal{J}_{n-1}+\mathcal{J}_{n+1}}{2}\\ 0 & 0 & \frac{-\mathcal{J}_{n-1}+\mathcal{J}_{n+1}}{2}\\ 0 & 0 & 0\end{bmatrix}, \tag{20.d}$$

$$\mathfrak{I}_{n,\cdot\cdot 13}(a)=\mathfrak{I}^T_{n,\cdot\cdot 31}(a)=\begin{bmatrix}0 & 0 & \frac{-\mathcal{J}_{n-1}+\mathcal{J}_{n+1}}{2}\\ 0 & 0 & \frac{\mathcal{J}_{n-1}+\mathcal{J}_{n+1}}{2}\\ 0 & 0 & 0\end{bmatrix}, \tag{20.e}$$

$$\mathfrak{I}_{n,\cdot\cdot 12}(a)=\mathfrak{I}^T_{n,\cdot\cdot 21}(a)=\begin{bmatrix}\frac{\mathcal{J}_{n-2}-\mathcal{J}_{n+2}}{4} & \frac{-\mathcal{J}_{n-2}+2\mathcal{J}_n-\mathcal{J}_{n+2}}{4} & 0\\ \frac{-\mathcal{J}_{n-2}-2\mathcal{J}_n-\mathcal{J}_{n+2}}{4} & \frac{-\mathcal{J}_{n-2}+\mathcal{J}_{n+2}}{4} & 0\\ 0 & 0 & 0\end{bmatrix}. \tag{20.f}$$

3 Application to elastodynamics

Let us apply now the THT to the displacement field $\mathbf{u}$ and the stress tensor σ in an isotropic elastic medium of mass density ρ, longitudinal and transverse velocities c_L , c_T, submitted to an external force density $\mathbf{f}$.

3.1 Formulation in the $(\,k\,,z\,,\omega\,)$ domain

Because the stress tensor is symmetric, the matrix Hankel transform can be rewritten by using Voigt notation:

$$\sigma_n^{\perp,\vdash}(r,z,\omega)=\begin{bmatrix}\sigma_{n,rr}^{\perp,\vdash}\\ \sigma_{n,\theta\theta}^{\perp,\vdash}\\ \sigma_{n,zz}^{\perp,\vdash}\\ \sigma_{n,\theta z}^{\perp,\vdash}\\ \sigma_{n,rz}^{\perp,\vdash}\\ \sigma_{n,r\theta}^{\perp,\vdash}\end{bmatrix}=\int_0^{+\infty}\mathbf{J}_n(k\,r)\begin{bmatrix}\Sigma_{n,kk}^{\perp,\vdash}(k,z,\omega)\\ \Sigma_{n,\alpha\alpha}^{\perp,\vdash}(k,z,\omega)\\ \Sigma_{n,zz}^{\perp,\vdash}(k,z,\omega)\\ \Sigma_{n,\alpha z}^{\perp,\vdash}(k,z,\omega)\\ \Sigma_{n,kz}^{\perp,\vdash}(k,z,\omega)\\ \Sigma_{n,k\alpha}^{\perp,\vdash}(k,z,\omega)\end{bmatrix}k\,\mathrm{d}k\,, \tag{21}$$

the (non-symmetric) matrix $\mathbf{J}_n(a)$ being:

$$\begin{bmatrix}\frac{-\mathcal{J}_{n-2}+2\mathcal{J}_n-\mathcal{J}_{n+2}}{4} & \frac{\mathcal{J}_{n-2}+2\mathcal{J}_n+\mathcal{J}_{n+2}}{4} & 0 & 0 & 0 & \frac{\mathcal{J}_{n-2}-\mathcal{J}_{n+2}}{2}\\ \frac{\mathcal{J}_{n-2}+2\mathcal{J}_n+\mathcal{J}_{n+2}}{4} & \frac{-\mathcal{J}_{n-2}+2\mathcal{J}_n-\mathcal{J}_{n+2}}{4} & 0 & 0 & 0 & \frac{-\mathcal{J}_{n-2}+\mathcal{J}_{n+2}}{2}\\ 0 & 0 & \mathcal{J}_n & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{-\mathcal{J}_{n-1}+\mathcal{J}_{n+1}}{2} & \frac{\mathcal{J}_{n-1}+\mathcal{J}_{n+1}}{2} & 0\\ 0 & 0 & 0 & \frac{\mathcal{J}_{n-1}+\mathcal{J}_{n+1}}{2} & \frac{-\mathcal{J}_{n-1}+\mathcal{J}_{n+1}}{2} & 0\\ \frac{\mathcal{J}_{n-2}-\mathcal{J}_{n+2}}{4} & \frac{-\mathcal{J}_{n-2}+\mathcal{J}_{n+2}}{4} & 0 & 0 & 0 & \frac{-\mathcal{J}_{n-2}-\mathcal{J}_{n+2}}{2}\end{bmatrix}. \tag{22}$$

One can demonstrate that, by using the THT, the *Hooke's law* (23) and the *Newton's second law* (24) (*e.g.*, [9] [10]) can be simply expressed in the $(\,k\,,z\,,\omega\,)$-domain:

$$\frac{1}{\rho}\mathbf{\Sigma}_n^{\perp,\vdash}=\begin{pmatrix}0 & 0 & c_L^2-2c_T^2\\ 0 & 0 & c_L^2-2c_T^2\\ 0 & 0 & c_L^2\\ 0 & c_T^2 & 0\\ c_T^2 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\partial_z\mathbf{U}_n^{\perp,\vdash}+k\begin{pmatrix}c_L^2 & 0 & 0\\ c_L^2-2c_T^2 & 0 & 0\\ c_L^2-2c_T^2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -c_T^2\\ 0 & c_T^2 & 0\end{pmatrix}\mathbf{U}_n^{\perp,\vdash}, \tag{23}$$

$$-\,\rho\omega^2\,\mathbf{U}_n^{\perp,\vdash}=\begin{pmatrix}0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&1&0&0&0\end{pmatrix}\partial_z\mathbf{\Sigma}_n^{\perp,\vdash}+k\begin{pmatrix}-1&0&0&0&0&0\\0&0&0&0&0&-1\\0&0&0&0&1&0\end{pmatrix}\mathbf{\Sigma}_n^{\perp,\vdash}+\mathbf{F}_n^{\perp,\vdash}. \tag{24}$$

3.2 The standard 2D problem

Note that (23) and (24) are independent from n and, remarkably, exactly the same as for the standard 2D problem (y-invariant) in Cartesian coordinates, " (k,z,ω)" being omitted in $\mathbf{U}_{2D}^{\vdash,\perp}(k,z,\omega)$ and $\mathbf{\Sigma}_{2D}^{\vdash,\perp}(k,z,\omega)$:

$$\mathbf{u}(x,z,\omega)=\int_0^{+\infty}\begin{pmatrix}\sin kx\\ \cos kx\\ \cos kx\end{pmatrix}\star\mathbf{U}_{2D}^{\vdash}+\begin{pmatrix}-\cos kx\\ \sin kx\\ \sin kx\end{pmatrix}\star\mathbf{U}_{2D}^{\perp}\,\mathrm{d}k\,, \tag{25}$$

and

$$\sigma(x,z,\omega)=\int_0^{+\infty}\begin{pmatrix}\cos kx\\ \cos kx\\ \cos kx\\ \cos kx\\ \sin kx\\ -\sin kx\end{pmatrix}\star\mathbf{\Sigma}_{2D}^{\vdash}+\begin{pmatrix}\sin kx\\ \sin kx\\ \sin kx\\ \sin kx\\ -\cos kx\\ \cos kx\end{pmatrix}\star\mathbf{\Sigma}_{2D}^{\perp}\,\mathrm{d}k\,. \tag{26}$$

3.3 Solutions

In any case, (23) and (24) leads to the ordinary differential system ($_{2D}$, n, $\vdash$ and $\perp$ omitted):

$$\begin{cases} c_T^2\,\partial_{z^2}U_k \;-k\,(c_L^2-c_T^2)\,\partial_z U_z + (\omega^2-k^2c_L^2)\,U_k &= \frac{-1}{\rho}F_k \\ c_T^2\,\partial_{z^2}U_{\alpha,y} & + (\omega^2-k^2c_T^2)\,U_{\alpha,y} = \frac{-1}{\rho}F_{\alpha,y} \\ c_L^2\,\partial_{z^2}U_z \;+k\,(c_L^2-c_T^2)\,\partial_z U_k + (\omega^2-k^2c_T^2)\,U_z &= \frac{-1}{\rho}F_z \quad . \end{cases} \tag{27}$$

The solutions of (27) without source-term are:

$$\mathbf{U}_P^{\pm}(k,z,\omega) = a_P^{\pm}(k,\omega)\,\mathbb{e}^{\mp\mathbb{i}\,k_{z,L}(k,\omega)\,z}\begin{pmatrix} k \\ 0 \\ \pm\mathbb{i}\,k_{z,L}(k,\omega)\end{pmatrix}, \tag{28}$$

$$\mathbf{U}_{SV}^{\pm}(k,z,\omega) = a_{SV}^{\pm}(k,\omega)\,\mathbb{e}^{\mp\mathbb{i}\,k_{z,T}(k,\omega)\,z}\begin{pmatrix} \pm\mathbb{i}\,k_{z,T}(k,\omega) \\ 0 \\ k\end{pmatrix}, \tag{29}$$

$$\mathbf{U}_{SH}^{\pm}(k,z,\omega) = a_{SH}^{\pm}(k,\omega)\,\mathbb{e}^{\mp\mathbb{i}\,k_{z,T}(k,\omega)\,z}\begin{pmatrix} 0 \\ k \\ 0\end{pmatrix}, \tag{30}$$

where the axial wavenumber is:

$$k_{z,X}(k,\omega) = \begin{cases} \operatorname{sign}(\omega)\sqrt{\dfrac{\omega^2}{c_X^2}-k^2} \;, & k < c_X\,|\omega| \\ -\mathbb{i}\,\sqrt{k^2-\dfrac{\omega^2}{c_X^2}} \;, & k > c_X\,|\omega| \end{cases} \quad . \tag{31}$$

The latter solutions correspond in the 2D case to the standard P, SV and SH waves, respectively. In the initial problem, for each azimuthal wavenumber n and for each azimuthal direction $\vdash$ or $\perp$, equivalent waves are defined and should be worthy of thorough study. Thus, they are used below to calculate the response of the half-space ($z \geqslant 0$) to a surface force-source $\mathbf{f}_0(\mathbf{x},\omega)$ located on the boundary ($z=0$).

3.4 Response of a half-space to a surface force-source

After an azimuthal Fourier series expansion and a THT, the problem to solve in the $(k\,,z\,,\omega)$-domain is to find the coefficients $a_{n,P}^{\vdash,\perp}$, $a_{n,SV}^{\vdash,\perp}$, $a_{n,SH}^{\vdash,\perp}$ such that the THT of the displacement $\mathbf{u}_n^{\vdash,\perp}(r,z,\omega)$ (defined in (10)) is rewritten as follows:

$$\mathbf{U}_n^{\vdash,\perp}(k,z,\omega) = \begin{pmatrix} k\,\mathbb{e}^{-\mathbb{i}\,k_{z,L}\,z} & \mathbb{i}\,k_{z,T}\,\mathbb{e}^{-\mathbb{i}\,k_{z,T}\,z} & 0 \\ 0 & 0 & k\,\mathbb{e}^{-\mathbb{i}\,k_{z,T}\,z} \\ \mathbb{i}\,k_{z,L}\,\mathbb{e}^{-\mathbb{i}\,k_{z,L}\,z} & k\,\mathbb{e}^{-\mathbb{i}\,k_{z,T}\,z} & 0 \end{pmatrix}\begin{pmatrix} a_{n,P}^{\vdash,\perp} \\ a_{n,SV}^{\vdash,\perp} \\ a_{n,SH}^{\vdash,\perp}\end{pmatrix}, \tag{32}$$

and that the boundary condition:

$$\mathbf{F}_{0,n}^{\vdash,\perp}(k,\omega) = \rho\,c_T^2\begin{pmatrix} 2\,\mathbb{i}\,k\,k_{z,L} & k^2-k_{z,T}^2 & 0 \\ 0 & 0 & \mathbb{i}\,k\,k_{z,T} \\ k^2-k_{z,T}^2 & 2\,\mathbb{i}\,k\,k_{z,T} & 0 \end{pmatrix}\begin{pmatrix} a_{n,P}^{\vdash,\perp} \\ a_{n,SV}^{\vdash,\perp} \\ a_{n,SH}^{\vdash,\perp}\end{pmatrix}. \tag{33}$$

is satisfied. This problem is easy to solve if the determinant of the square matrix in (33) is different from zero, *i.e.* the Rayleigh waves are not excited: $F_{0,n}^{\vdash,\perp}(k,\pm c_R k) = \vec{0}$, c_R being the Rayleigh-wave velocity (*e.g.*, [9] [10]).

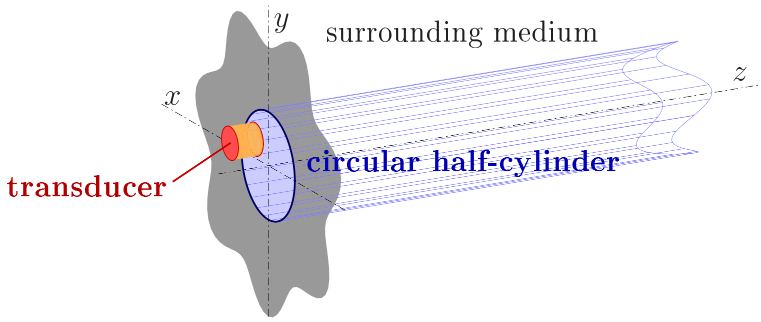

Figure 2: A half-cylinder surrounded by a half-infinite elastic medium and excited at its end.

3.5 Case of a cylinder embedded in an elastic media

The previous section summarizes the first step to calculate, by using a standard scattering approach, the field generated into a circular half-cylinder surrounded by a half-infinite elastic medium by a surface force-source located at its end (see. Fig.2).

Indeed, the incident field in the cylindrical waveguide is given by (32) and (33). In a second step, this incident field is expressed in the (r, k_z, ω)-domain to calculate the reflected field (guided wave) and the transmitted field (radiated into the surrounded medium). More details are given in a related paper [7].

Consequently, the THT is relevant for modeling circular problems as the latter because each case characterized by an azimuthal wavenumber n and an azimuthal direction $\vdash$ or $\perp$ can be treated separately and is similar to the standard 2D problem.

4 References

[1] R. Bracewell, *The Fourier transform and its applications*, 3rd edition, McGraw-Hill Science Engineering, 1st edition in 1965, New York, 1986, ISBN 978-0-07-007015-8.

[2] I. N. Sneddon, Z. Olesiak, W. Nowacki and G. Eason, *Application of integral transforms in the theory of elasticity*, Springer-Verlag, Wien, New York, 1975.

[3] W. C. Chew and J. A. Kong, “Resonance of nonaxial symmetric modes in circular microstrip disk antenna,” J. Math. Phys., vol. 21, pp. 2590–2598, 1980.

[4] S. M. Ali, W. C. Chew and J. A. Kong, “Vector Hankel transform analysis of annular-ring microstrip antenna,” IEEE Transactions on Antennas and Propagation, vol. 30, pp. 637–644, 1982.

[5] W. C. Chew and T. M. Habashy, “The use of vector transforms in solving some electromagnetic scattering problems,” IEEE Transactions on Antennas and Propagation, vol. 34, pp. 871–879, 1986.

[6] E. Kausel, *Fundamental Solutions in Elastodynamics: A Compendium*, Cambridge University Press, 2006, ISBN 0-521-85570-5.

[7] S. Yaacoubi, E. Ducasse, M. Deschamps and L. Laguerre, “Calculation of the ultrasonic field generated by a transducer non-axially located at the end of a cylindrical waveguide surrounded by an elastic medium,” in “ICU’09, Santiago of Chile,” 2009.

[8] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th edition, Cambridge University Press, New York, 1927, ISBN 978-0-521-09189-3.

[9] F. J. Fedorov, *Theory of elastic waves in crystals*, Plenum Press / Kluwer Academic Publishers, New York, 1968, ISBN 0-30630-309-4.

[10] B. A. Auld, *Acoustic Fields and Waves in Solids*, 2nd edition, Krieger Pub Co, 1990, ISBN 0-89874-782-1.