A posteriori error estimation for stochastic static problems

D. H. Mac[1], S. Clénet[1]
[1]L2EP, Arts et Métiers ParisTech, 8 bd de Louis XIV, 59000 Lille, France

Abstract—To solve stochastic static field problems, a discretization by the Finite Element Method can be used. A system of equations is obtained with the unknowns (scalar potential at nodes for example) being random variables. To solve this stochastic system, the random variables can be approximated in a finite dimension functional space - a truncated polynomial chaos expansion. The error between the exact solution and the approximated one depends not only on the spatial mesh but also on the discretization along the stochastic dimension. In this paper, we propose an *a posteriori* estimation of the error due to the discretization along the stochastic dimension.

Index Terms—Stochastic problems, Finite Element Method, Polynomial Chaos Expansion, Magnetostatics, Error estimation.

I. Introduction

Numerical modelling can be used to predict the behavior of an electromagnetic device. The Maxwell equations can be solved using the Finite Element Method (FEM). In a linear static problem, FEM leads to a linear system of equations. The unknowns could be the value of the scalar potential at the nodes or the circulation of the vector potential at the edges, for example.

When the input data (behavior laws of the material, the geometry of the device, ...) of the numerical model are uncertain due to several factors like the ageing of the material, the imperfections of the manufacturing processes, ..., the Maxwell equations become stochastic. Consequently, system of equations obtained by FEM becomes also stochastic. The unknowns are random variables. Sampling methods like the Monte Carlo Simulation Method (MCSM) can be used to estimate statistical moments or probability of failures. Another method consists of approximating the solution of the system of equations given by FEM along the random dimension. In engineering, this approach was introduced by Ghanem [1] in mechanics and widely developed in this area. In computational electromagnetic, this approach was introduced in [2] and [3].

The Polynomial Chaos Expansion (PCE) [4] is one of the most popular methods to approximate random variables. The numerical solution can be obtained then by two discretization steps, one along the spatial dimension (mesh) and the other along the random dimension using a truncated PCE. To evaluate the numerical error (i.e. the distance between the numerical solution and the exact solution), one can distinguish two kinds of error estimation, *a priori* and *a posteriori* error estimations. In this paper, we are interested in *a posteriori* error estimation which is calculated from the numerical solution and so, the error estimation is evaluated after the numerical solution of

the problem. The *a posteriori* error estimation of a stochastic problem has been already addressed in literature [5], [6], [7]. In [5], an error estimation based on the hyper-circle theorem is proposed. This error requires the solutions of two complementary formulations. This error estimator takes simultaneously into account the error due to the discretizations along the spatial and the random dimensions, giving a so-called "global estimation". In [6], an error estimation based on the solution of an adjoint problem is proposed. The estimator is applied to some non-linear problems where the adjoint problem becomes linear and the solution of the adjoint problem needs then less time than the initial one. The error estimation in [6] is also global and can be applied only for the numerical solution obtained by the Spectral Stochastic Finite Element Method (SSFEM) [2], [3]. In [7], an error estimator enabling us to evaluate the error due the stochastic discretization (only due to PCE), the so-called stochastic error, has been proposed. This estimator is based on the solution of the error equation where the right hand side is the residual evaluated from the numerical solution. The error equation has to be solved using a PCE with higher order than the one used for the numerical solution of the initial equation. In this method, two stochastic problems have to be solved.

In this paper, we propose a stochastic error estimator that is evaluated directly from the residual calculated from the numerical solution. The proposed error estimation is tested on an academic example.

II. Magnetostatic problem with uncertainties on the behavior law

We are interested in a magnetostatic problem defined on a domain D with an uncertain permeability. The permeability can be modeled by a random field $\mu(x, \theta)$ where the parameter θ refers to an elementary event and x the position. We assume that the random field $\mu(x, \theta)$ can be expressed explicitly (or at least approximated by a Karhunen−Loeve expansion [8]) as a function of a random vector $\boldsymbol{\xi}(\theta) = (\xi_1(\theta), \xi_2(\theta), ..., \xi_M(\theta))$ where $\xi_1(\theta), \xi_2(\theta), ..., \xi_M(\theta)$ are real independent random variables with known probability density functions. We denote $\Theta^M \subset \mathbf{R}^M$ the domain of the random vector $\boldsymbol{\xi}(\theta)$. In the following, to simplify the notations, the dependency on θ of the random vector $\boldsymbol{\xi}$ will be removed. We assume also that the permeability is bounded, meaning that:

$$0 < \mu_{min}(x) \leq \mu(x, \boldsymbol{\xi}) \leq \mu_{max}(x) < \infty \quad \forall \boldsymbol{\xi} \subset \Theta^M. \quad (1)$$

The equations of the magnetostatic problem defined on D, can be written as:

$$\begin{cases} \nabla \cdot \mathbf{B}(x,\xi) = 0 \\ \nabla \times \mathbf{H}(x,\xi) = \mathbf{J}_s(x) \\ \mathbf{B}(x,\xi) = \mu(x,\xi)\,\mathbf{H}(x,\xi) \end{cases} \tag{2}$$

with $x \in D$, $\boldsymbol{\xi} \in \Theta^M$, where $\mathbf{H}$ and $\mathbf{B}$ are the magnetic field and the magnetic flux density, respectively. The source term $\mathbf{J}_s$, being divergence free, can be written as $\mathbf{J}_s(x) = \nabla \times \mathbf{H}_s(x)$. For the sake of simplicity, homogeneous boundary conditions are prescribed:

$$\mathbf{B}(x,\xi) \cdot \mathbf{n}(x) = 0 \quad \forall x \in \Gamma_D, \forall \boldsymbol{\xi} \in \Theta^M. \tag{3}$$

We should mention that the results proposed in this paper can be extended also to the case with the mixed boundary conditions, i.e $\mathbf{B}(x,\xi)\cdot\mathbf{n}(x) = 0$ on one part of Γ_D and $\mathbf{H}(x,\xi)\times \mathbf{n}(x) = 0$ on the complementary part of the boundary Γ_D. To solve the problem described in (2) and (3), the scalar potential Ω can be introduced such that:

$$\mathbf{H}(x,\xi) = -\nabla\Omega(x,\xi) + \mathbf{H}_s(x)$$

or the vector potential $\mathbf{A}$, such that:

$$\mathbf{B}(x,\xi) = \nabla \times \mathbf{A}(x,\boldsymbol{\xi}).$$

In this paper, we will focus on the scalar potential formulation. The extension to the vector potential formulation case is straightforward. The scalar potential formulation yields the following weak form:

$$\int_D \mu(x,\xi)\,\nabla\Omega(x,\boldsymbol{\xi}) \cdot \nabla\lambda(x)\,dx =$$

$$\int_D \mu(x,\xi)\,\mathbf{H}_s(x) \cdot \nabla\lambda(x)\,dx \quad \forall \lambda \in H^1(D), \forall \boldsymbol{\xi} \in \Theta^M \tag{4}$$

where $H^1(D)$ is a functional space defined by:

$$H^1(D) = \left\{ v \in L^2(D) \;\middle|\; \int_D |\nabla v(x)|^2\; dx < \infty \right\}. \tag{5}$$

Clearly, the solution of (4) is defined up to an additive constant. However, one can notice that the magnetic field is unique and does not depend on the additive constant. In a general case, the analytical solution of (4) is not reachable. Thus, we introduce a tetrahedral mesh $\mathcal{M}$ of the domain D with n_0 nodes, n_1 edges, n_2 facets and n_3 elements. An approximation Ω_h of the scalar potential Ω is sought in $V_h^{i_0}$ such that

$$\int_D \mu(x,\xi)\,\nabla\Omega_h(x,\boldsymbol{\xi}) \cdot \nabla w(x)\,dx =$$

$$\int_D \mu(x,\xi)\,\mathbf{H}_s(x) \cdot \nabla w(x)\,dx \quad \forall w \in V_h^{i_0}, \forall \boldsymbol{\xi} \in \Theta^M \tag{6}$$

where the discrete functional space $V_h^{i_0}$ is defined by:

$$V_h^{i_0} = \mathrm{span}\,\{w_{0i}\,|\,i = 1,2,...,i_0-1,i_0+1,...n_0\} \tag{7}$$

with w_{0i} the shape function [9] associated with the node i. One can notice that the choice of the discrete functional space

$V_h^{i_0}$ imposes a gauge condition such that the scalar potential is equal to zero at the node i_0. This gauge condition imposes the uniqueness of the solution of (6). The other gauge conditions [13] can be implemented but this has no consequence on our proposed method.

III. POLYNOMIAL CHAOS EXPANSION

The solution of (6) can be written under the form

$$\Omega_h(x,\boldsymbol{\xi}) = \sum_{i=1,i\neq i_0}^{n_0} \Omega_h^i(\boldsymbol{\xi})\,w_{0i}(x) \tag{8}$$

with $\Omega_h^i(\boldsymbol{\xi})$ the value of the scalar potential at the node i that is a random variable. To obtain an explicit expression of $\Omega_h^i(\boldsymbol{\xi})$, the idea is to approximate it in a finite dimension functional space. A truncated PCE can be an appropriate choice if the variation of $\Omega_h^i(\boldsymbol{\xi})$ as a function of $\boldsymbol{\xi}$ is quite smooth. The scalar potential $\Omega_h(\boldsymbol{\xi})$ at the node i is then approximated by:

$$\Omega_h^i(\boldsymbol{\xi}) \approx \sum_{j=0}^{P} \Omega_{ij}\Psi_j(\boldsymbol{\xi}) \tag{9}$$

where Ψ_j, $j = 0 : P$ is a set of orthonormal polynomials [4] and Ω_{ij} are the real coefficients to be determined. The numerical solution of the problem can be written as:

$$\Omega_{h,P}(x,\boldsymbol{\xi}) = \sum_{j=0}^{P} \sum_{i=1,i\neq i_0}^{n_0} \Omega_{ij}\,w_{0i}(x)\Psi_j(\boldsymbol{\xi}). \tag{10}$$

Two categories of methods, non intrusive and intrusive, have been proposed in the literature to determine Ω_{ij}. In the non-intrusive methods [10], [11], the deterministic FEM model can be directly used to obtain the stochastic solution. Indeed, the coefficients Ω_{ij} are determined by:

$$\Omega_{ij} = \sum_{k=1}^{Q} \Omega_h^i(\boldsymbol{\xi}_k)\Psi_j(\boldsymbol{\xi}_k)\omega_k. \tag{11}$$

The evaluation of coefficients Ω_{ij} in (11) requires Q evaluations $\Omega_h^i(\boldsymbol{\xi}_k)$, for $k = 1 : Q$, where $\Omega_h^i(\boldsymbol{\xi}_k)$ refers to the scalar potential Ω_h at node i at well fitted points $\boldsymbol{\xi}_k$. Thus, Q deterministic problems (6) corresponding to Q realizations of the permeability $\mu(x,\boldsymbol{\xi}_k)$ have to be solved. Several choices of the weights ω_k and of the points $\boldsymbol{\xi}_k$ are possible (see [10], [11]). For a given mesh, the numerical solution (10) depends on the choice of the set of polynomials Ψ_j and on the set of the evaluation points $\boldsymbol{\xi}_k$ and on the associated weights ω_k.

In the intrusive method (SSFEM method [2], [3]), the coefficients Ω_{ij} are determined by using the Galerkin projection:

$$\mathrm{E}\left(\int_D \mu(x,\xi)\,\nabla\Omega_{h,P}(x,\boldsymbol{\xi}) \cdot \nabla w_{0i}(x)\,dx\,\Psi_j(\boldsymbol{\xi})\right) =$$

$$\mathrm{E}\left(\int_D \mu(x,\xi)\,\mathbf{H}_s(x) \cdot \nabla w_{0i}(x)\,dx\,\Psi_j(\boldsymbol{\xi})\right) \tag{12}$$

with $i = 1 : n_0\backslash i_0, j = 0 : P$ and $\mathrm{E}(X(\boldsymbol{\xi}))$ denoting the expectation of the random variable $X(\boldsymbol{\xi})$. Equation (12) leads to a linear matrix system of dimension $(n_0-1)\times(P+1)$ where the solution is the vector of the coefficients Ω_{ij}. For a given mesh, the numerical solution (10) depends on the choice of

the set of polynomials and on the accuracy of the solution of the linear system given by (12).

IV. Stochastic error estimation

A. Definition of the numerical errors

The stochastic error e_{sto} is defined by the following

$$e_{sto}^2(\xi) = \int_D \mu(x, \xi) \nabla (\Omega_{h,P}(x, \xi) - \Omega_h(x, \xi)) \cdot$$
$$\nabla (\Omega_{h,P}(X, \xi) - \Omega_h(x, \xi)) \, dx. \qquad (13)$$

Using the same norm, the spatial error e_{spa} and the global error e_{glo} can also be defined respectively as the distance between Ω_h and Ω and the distance between $\Omega_{h,P}$ and Ω. From (4), (6) and (10) it can be shown that

$$e_{glo}^2(\xi) = e_{sto}^2(\xi) + e_{spa}^2(\xi). \qquad (14)$$

The global error estimation requires estimations of both the stochastic error e_{sto} and the spatial error e_{spa}. In the following, we will focus only on the estimation of the stochastic error e_{sto} which is directly linked to the discretization of Ω_h along the stochastic dimension.

B. Stochastic error estimator

We propose the following estimator for $e_{sto}^2(\xi)$

$$\eta_{sto}^2(\xi) = \mathbf{R}^t(\xi) \mathbf{\Lambda}_0^{-1} \mathbf{R}(\xi) \qquad (15)$$

where $\mathbf{R}$ is a stochastic residual vector whose coefficients are given by:

$$[\mathbf{R}]_i(\xi) = \int_D \mu(x, \xi) \nabla \Omega_{h,P}(x, \xi) \cdot \nabla w_{0i}(x) \, dx -$$
$$\int_D \mu(x, \xi) \nabla \mathbf{H}_s(x) \cdot \nabla w_{0i}(x) \, dx \qquad (16)$$

with $i = 1 : n_0 \backslash i_0$ and $\mathbf{\Lambda}_0$ the expectation of the stiffness matrix defined by:

$$[\mathbf{\Lambda}_0]_{ij} = \int_D E(\mu(x, \xi)) \nabla w_{0i}(x) \cdot \nabla w_{0j}(x) \, dx \qquad (17)$$

with $i = 1 : n_0 \backslash i_0$, $j = 1 : n_0 \backslash i_0$. It can be shown that η_{sto} in (15) is an equivalent measure of the stochastic error e_{sto} meaning that

$$k_1 \eta_{sto}^2(\xi) \leq e_{sto}^2(\xi) \leq k_2 \eta_{sto}^2(\xi) \qquad (18)$$

with k_1 and k_2 are positive coefficients and independent of Ω_h and of $\Omega_{h,P}$. In particular, we can deduce an explicit expression of these coefficients:

$$k_1 = \min_{x \in D} \left(\frac{E(\mu(x, \xi))}{\mu_{max}(x)} \right) \qquad (19)$$

$$k_2 = \max_{x \in D} \left(\frac{E(\mu(x, \xi))}{\mu_{min}(x)} \right)$$

(see the APPENDIX for the proof of (18)). Due to the fact that $\eta_{sto}^2(\xi) \geq 0$, $\forall \xi \subset \Theta^M$, and from (18), the expectation $E(\eta_{sto}^2(\xi))$ could be an appropriate indicator to quantify the

stochastic error e_{sto}. To evaluate $E(\eta_{sto}^2(\xi))$, the residual $\mathbf{R}$ is deduced from (16) as shown:

$$\mathbf{R}(\xi) = \sum_{j=0}^{P_1} \mathbf{R}_j \Psi_j(\xi) \qquad (20)$$

where $\mathbf{R}_j$ are real vectors. Due to the fact that $\Psi_j(\xi)$ are orthonormal, we obtain:

$$E(\eta_{sto}^2(\xi)) = \sum_{j=0}^{P_1} \mathbf{R}_j^t \mathbf{\Lambda}_0^{-1} \mathbf{R}_j. \qquad (21)$$

From (19), the ratio between the upper and the lower bounds of the stochastic error (18) can be evaluated. In practice, this ratio corresponds to a small number of units. Furthermore, the estimation (18) is independent of the method used to solve the stochastic problem (SSFEM, Non intrusive, etc.) and can be extended to the different stochastic approximation bases (truncated PCE in this paper or wavelet decomposition [12], etc.).

V. Numerical example

We are interested in a magnetostatic problem presented in Fig. 1. The domain is divided into 5 sub-domains with the
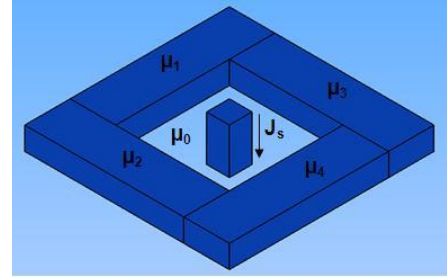

Figure 1: Magnetostatic problem defined in the domain D.

relative permeabilities $\mu_0 = 1$, $\mu_1 = \mu_2 = 1000$. μ_3 and μ_4 are two independent uniform random variables defined on $[600 - 1400]$. The current $\mathbf{J}_s$ is imposed equal to 1A. We apply the SSFEM method [2] to solve this problem. For a given numerical solution $\Omega_{h,P}(x, \xi)$, the mean value of the stochastic estimator η_{sto}^2 is compared to the mean value of the stochastic error e_{sto}^2 estimated by the MCSM described on the flow chart in Fig. 2. As mentioned in Section III, with a given mesh, the
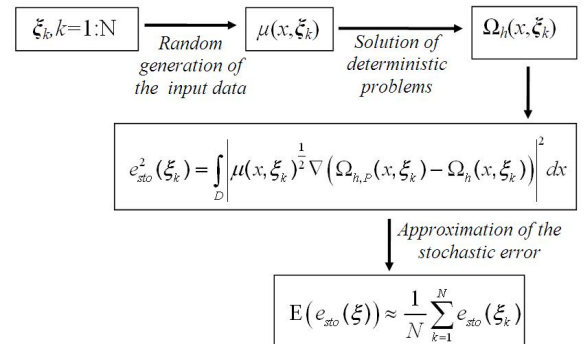

Figure 2: Monte-Carlo method.

numerical solution (10) depends on the choice of the set of polynomials and on the accuracy of the solution of the SSFEM system of equations given by (12). In this example, a mesh with 2617 nodes and the full tensorised Legendre PCE [4] are used. The set of the polynomials depends then on the order of truncation p. The SSFEM linear system of equations is solved using an iterative method and the accuracy of the solution is fixed by a stopping criterion ε based on a residual norm.

The evolution of the mean value of the stochastic estimator and the mean value of stochastic error estimated by MCSM as a function of the order p of PCE and of the stopping criterion ε is presented in Fig. 3.
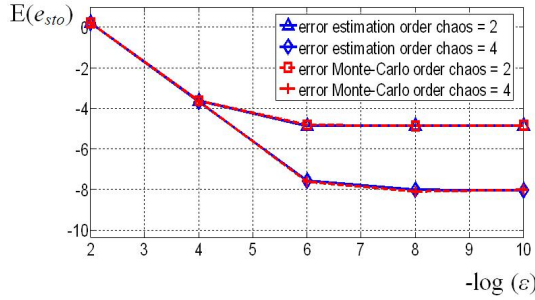

Figure 3: Evolution of the mean value of the estimator and of the MCSM estimated stochastic error.

According to Fig. 3, we can deduce that : -The estimator and the stochastic error estimated by the MSCM are very close. -While the accuracy level of the solution of the SSFEM matrix system is low (ε is greater than 10^{-4} in Fig. 3) the stochastic error is the same with different orders of PCE. -While the accuracy level is quite high (ε is lower than 10^{-4} in Fig. 3) a higher order of PCE yields a smaller stochastic error. -With a given order of PCE, when the accuracy level increases, the evolution of the stochastic error decreases up to a given value before being stable ($\log(\varepsilon) = -6$ with order $p = 2$ and $\log(\varepsilon) = -8$ with order $p = 4$). It is thus wasteful to increase the accuracy level of the solution of the SSFEM matrix system beyond these points.

VI. Conclusion

We have presented a stochastic *a posteriori* error estimator for a stochastic magetostatic problem. The estimator allows us to evaluate the error due to the stochastic discretization (truncated PCE in this paper). The estimator is based on a residual of the discrete weak formulation. For a given mesh, by using the estimator, one can compare the accuracy of different numerical solutions. In this paper, we are interested only in the stochastic error. In order to obtain a complete analysis, the error due to the spatial discretization has to be taken into account.

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APPENDIX

(18) can be proved by using the two following lemmas.

Lemma 1. For $\Lambda(\xi)$ the stiffness matrix defined by:

$$[\Lambda]_{ij}(\xi) = \int_D \mu(x,\xi)\nabla w_{0i}(x) \cdot \nabla w_{0j}(x)\, dx \qquad (22)$$

with $i = 1 : n_0 \backslash i_0$, $j = 1 : n_0 \backslash i_0$, we have

$$e_{sto}^2(\xi) = \mathbf{R}^t(\xi)\Lambda^{-1}(\xi)\mathbf{R}(\xi) \qquad (23)$$

□

Lemma 2. For $\Lambda_1(\xi)$ and $\Lambda_2(\xi)$, two matrices of dimension $(n_0 - 1) \times (n_0 - 1)$ such that:

$$[\Lambda_1]_{ij}(\xi) = \int_D \mu_1(x,\xi)\nabla w_{0i}(x) \cdot \nabla w_{0j}(x)\, dx$$

$$[\Lambda_2]_{ij}(\xi) = \int_D \mu_2(x,\xi)\nabla w_{0i}(x) \cdot \nabla w_{0j}(x)\, dx$$

with $0 < \mu_1(x,\xi) \le \mu_2(x,\xi)$, $\forall x \in D, \forall \xi \in \Theta^M$. We thus obtain:

$$\mathbf{R}^t(\xi)\Lambda_2^{-1}(\xi)\mathbf{R}(\xi) \le \mathbf{R}^t(\xi)\Lambda_1^{-1}(\xi)\mathbf{R}(\xi), \ \forall \xi \in \Theta^M \qquad (24)$$

□

From lemma 1 and lemma 2, one can deduce (18) due to the fact that

$$\frac{1}{k_2}\mathrm{E}(\mu(x,\xi)) \le \mu(x,\xi) \le \frac{1}{k_1}\mathrm{E}(\mu(x,\xi))$$

$\forall x \in D, \forall \xi \in \Theta^M$.