Analysis of Intrinsic Mode Functions: A PDE Approach

El Hadji S. Diop, R. Alexandre, and A. O. Boudraa

Abstract—**The empirical mode decomposition is a powerful tool for signal processing. Because of its original algorithmic, recent works have contributed to its theoretical framework. Following these works, some mathematical contributions on its comprehension and formalism are provided. In this paper, the so called *local mean* is computed in such a way that it allows the use of differential calculus on envelopes. This new formulation makes us prove that iterations of the sifting process are well approximated by the resolution of partial differential equations (PDE). Intrinsic mode functions are originally defined in a intuitive way. Herein, a mathematical characterization of modes is given with the proposed PDE-based approach.**

Index Terms—**Empirical mode decomposition, intrinsic mode function, partial differential equations.**

I. Introduction

THE empirical mode decomposition (EMD) was introduced by Huang *et al.* [1] for analyzing non-stationary time series derived from linear and nonlinear systems. It consists in decomposing signals into several basic components called intrinsic mode functions (IMFs) or modes, and a residual understood as the signal trend. Any given signal is then decomposed by the EMD into a sum of IMFs, which are generated at each scale going from fine to coarse by an iterative procedure, the sifting process (SP). Many numerical simulations have been performed for a better understanding of the behavior of the EMD [2], [3]. However, a lack of a strong theoretical framework remains its main criticism. Recent works [4]–[8] were performed for a better understanding, though. The study of SP is difficult particularly because of the loose definition of the so-called *local mean* [1]. Indeed, it involves notions of upper and lower envelopes of the signal, which are not easy to handle for further calculus, whatever the interpolations one could use. Following Huang *et al.* [1], for a given signal denoted by $S(x)$, the EMD algorithm can be summarized as follows.

1) Find all the extrema of $S(x)$.
2) Interpolate the maxima of $S(x)$ (resp. the minima of $S(x)$), $E_{max}(x)$ (resp. $E_{min}(x)$).
3) Compute the *local mean*: $m(x) = (1/2)(E_{max}(x) + E_{min}(x))$.
4) Extract and sift the detail $f(x) = S(x) - m(x)$.
5) Iterate on the residual $m(x)$ (up to the absence of extrema).

Then, any signal $S(x)$ will be decomposed by the EMD as $S(x) = \sum_{k=1}^{N} f_k(x) + r(x)$, where f_k denotes the so-called k^{th} IMF, and $r(x)$ is the residual.

In this work, we slightly change the definition of the *local mean* by computing differently the local upper and lower envelopes as in [8]. The local mean is then calculated in a better way so that we can use differential calculus. In [8], we proposed a PDE formulation of the SP, but limited to a class of signals. Here, a more general framework is presented.

II. A PDE Formulation of the Sifting Process

Let $S : \Omega \to \mathbb{R}$, $S = S(t)$, $t \in \Omega$ be a continuous signal. Ω is an open bounded set of $\mathbb{R}$, and its boundary is denoted by $\partial\Omega$. The SP is fully determined by the sequence $(h_n)_{n\in\mathbb{N}}$ defined by [8]:

$$\begin{cases} h_{n+1} = h_n - \frac{1}{2}\left[\hat{h_n}^u + \hat{h_n}^l\right] \\ h_0 = S. \end{cases} \tag{1}$$

where $\hat{h_n}^u$ (resp. $\hat{h_n}^l$) denotes a fixed continuous interpolation of the maxima (resp. minima) of h_n. Let Φ be the operator defined on suitable functions h (for example $h \in C^0(\bar{\Omega})$) by $\Phi(h) = (1/2)[\hat{h}^u + \hat{h}^l]$. Let $I_d : \Omega \longrightarrow \mathbb{R}$ be the identity application. Then, $\forall n \in \mathbb{N}$, we have $h_n = (I_d - \Phi)^n h_0$, and therefore, by letting $\Psi = I_d - \Phi$, $\forall n \in \mathbb{N}$, we get $h_n = \Psi^n h_0$.

Let $\delta > 0$ be a small parameter.

Definition 1: For all $h \in L^\infty(\Omega)$, we define the operator m_δ, mapped from $L^\infty(\Omega)$ into $L^\infty(\Omega)$, by:

$$m_\delta h(x) = \frac{1}{2}\left[\sup_{|y|<\delta} h(x+y) + \inf_{|y|<\delta} h(x+y)\right], \ \forall x \in \Omega. \tag{2}$$

The main idea of the paper is to replace the local mean Φ by the operator m_δ.

Let's now define the following subsets of Ω, and depending on h:

$$\Omega_{h,1} = \{x \in \Omega \text{ such that } h''(x) = 0\} \tag{3}$$

$$\Omega_{h,2} = \{x \in \Omega \text{ such that } h''(x) \neq 0 \text{ and } |h'(x)| > \delta\,|h''(x)|\} \tag{4}$$

$$\Omega_{h,3} = \{x \in \Omega \text{ such that } h''(x) \neq 0 \text{ and } |h'(x)| \leq \delta\,|h''(x)|\}. \tag{5}$$

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As usual, we shall use the norm $\|h\|_\infty = \sup_{x\in\Omega}|h(x)|$ for any $h \in L^\infty(\Omega)$.

Theorem 1: Let $h \in C^3(\Omega)$ such that h''' is bounded on Ω. For $\delta > 0$ small, one has

$$m_\delta h(x) = h(x) + O(\delta^3)\|h'''\|_\infty, \text{ if } x \in \Omega_{h,1} \quad (6)$$

$$m_\delta h(x) = h(x) + \frac{\delta^2}{2}h''(x) + O(\delta^3)\|h'''\|_\infty, \text{ if } x\in\Omega_{h,2} \quad (7)$$

$$m_\delta h(x) = h(x) - \frac{1}{4}\frac{[h'(x)]^2}{h''(x)} + sign\,(h''(x))\,\frac{\delta}{2}\,|h'(x)| + \frac{\delta^2}{4}h''(x) + O(\delta^3)\|h'''\|_\infty, \text{ if } x \in \Omega_{h,3}. \quad (8)$$

Proof: See the Appendix.

We recall the sequence $(h_n)_n$ which holds for approximate and virtual IMFs $h_n = (I_d - \Phi)^n h_0$. Then, we define the sequence $(\bar{h}_n)_{n\in\mathbb{N}}$ as:

$$\begin{cases} \bar{h}_{n+1} = (I_d - m_\delta)\bar{h}_n \\ \bar{h}_0 = S, \text{ on } \Omega. \end{cases} \quad (9)$$

Thanks to Theorem 1, (9) is developed as follows.

- If $\bar{h}''_n(x) = 0$, then: $\bar{h}_{n+1}(x) = O(\delta^3)\|\bar{h}'''_n\|_\infty$.
- If $\bar{h}''_n(x) \neq 0$ and $|\bar{h}'_n(x)| > \delta|\bar{h}''_n(x)|$, then $\bar{h}_{n+1}(x) = -(\delta^2/2)\bar{h}''_n(x) + O(\delta^3)\|\bar{h}'''_n\|_\infty$.
- If $\bar{h}''_n(x) < 0$ and $|\bar{h}'_n(x)| \leq \delta|\bar{h}''_n(x)|$, then $\bar{h}_{n+1}(x) = -(\delta^2/4)\bar{h}''_n(x) + (\delta/2)|\bar{h}'_n(x)| + (1/4)([\bar{h}'_n(x)]^2/\bar{h}''_n(x)) + O(\delta^3)\|\bar{h}'''_n\|_\infty$.
- If $\bar{h}''_n(x) > 0$ and $|\bar{h}'_n(x)| \leq \delta\bar{h}''_n(x)$, then: $\bar{h}_{n+1}(x) = -(\delta^2/4)\bar{h}''_n(x) - (\delta/2)|\bar{h}'_n(x)| + (1/4)([\bar{h}'_n(x)]^2/\bar{h}''_n(x)) + O(\delta^3)\|\bar{h}'''_n\|_\infty$.

Let us next fix $\tau > 0$, τ is chosen small enough. Let $\tilde{h}$ given by

$$\begin{cases} \tilde{h}: & \Omega \times \{\tau\mathbb{N}\} \longrightarrow \mathbb{R} \\ & (x, n\tau) \longmapsto \bar{h}_n(x). \end{cases} \quad (10)$$

Let's consider a smoothed enough interpolation of $\tilde{h}$, also denoted, with some notation abuse, by $\tilde{h}$. Using a Taylor expansion, one has $\bar{h}_{n+1}(x) = \tilde{h}(x, n\tau + \tau) = \tilde{h}(x, n\tau) + \tau(\partial\tilde{h}/\partial t)(x, n\tau) + o(\tau^2)$. Thus, choosing $\tau = \delta^2$, yields the following.

- If $\bar{h}''_n(x) = 0$, then: $(\partial\tilde{h}/\partial t)(x, n\tau) = -(1/\delta^2)\tilde{h}(x, n\tau) + O(\delta^3)\|\bar{h}'''_n\|_\infty$.
- If $\bar{h}''_n(x) \neq 0$ and $|\bar{h}'_n(x)| > \delta|\bar{h}''_n(x)|$, then $(\partial\tilde{h}/\partial t)(x, n\tau) = -(1/\delta^2)\tilde{h}(x, n\tau) - (1/2)\tilde{h}''_n(x, n\tau) + O(\delta^3)\|\bar{h}'''_n\|_\infty$.
- If $\bar{h}''_n(x) < 0$ and $|\bar{h}'_n(x)| \leq \delta|\bar{h}''_n(x)|$, then: $(\partial\tilde{h}/\partial t)(x, n\tau) = -(1/\delta^2)\tilde{h}(x, n\tau) - (1/4)\tilde{h}''(x, n\tau) + (1/2\delta)|\tilde{h}'(x, n\tau)| + (1/4\delta^2)([\tilde{h}'(x, n\tau)]^2/\tilde{h}''(x, n\tau)) + O(\delta^3)\|\bar{h}'''_n\|_\infty$.
- If $\bar{h}''_n(x) > 0$ and $|\bar{h}'_n(x)| \leq \delta|\bar{h}''_n(x)|$, then $(\partial\tilde{h}/\partial t)(x, n\tau) = -(1/\delta^2)\tilde{h}(x, n\tau) - (1/4)\tilde{h}''(x) - (1/2\delta)|\tilde{h}'(x, n\tau)| + (1/4\delta^2)([\tilde{h}'(x, n\tau)]^2/\tilde{h}''(x, n\tau)) + O(\delta^3)\|\bar{h}'''_n\|_\infty$.

In view of the above computations, it is therefore natural to introduce for any smooth and fixed function $h : \Omega \times \mathbb{R} \to \mathbb{R}$, $h = h(x, t)$, the following subsets:

$$\Omega_1(h) = \left\{(x,t) \in \Omega \times \mathbb{R} \text{ s.t. } \frac{\partial^2 h}{\partial x^2}(x,t) = 0\right\} \quad (11)$$

$$\Omega_2(h) = (x,t) \in \Omega \times \mathbb{R} \text{ s.t. } \frac{\partial^2 h}{\partial x^2}(x,t) \neq 0 \text{ and } \left|\frac{\partial h}{\partial x}(x,t)\right| > \delta\left|\frac{\partial^2 h}{\partial x^2}(x,t)\right| \quad (12)$$

$$\Omega_3(h) = (x,t) \in \Omega \times \mathbb{R} \text{ s.t. } \frac{\partial^2 h}{\partial x^2}(x,t) \neq 0 \text{ and } \left|\frac{\partial h}{\partial x}(x,t)\right| \leq \delta\left|\frac{\partial^2 h}{\partial x^2}(x,t)\right|. \quad (13)$$

Definition 2: A $p\,\delta - IMF$ is a function k defined as:

$$\begin{cases} k: & \Omega \longrightarrow \mathbb{R} \\ & x \longmapsto k(x) = h(x, t_0), \end{cases} \quad (14)$$

for some $t_0 > 0$, where h is solution of the free-boundary PDE:

$$\begin{cases} \frac{\partial h}{\partial t} + \frac{1}{\delta^2}h = 0, & \text{on } \Omega_1(h) \\ \frac{\partial h}{\partial t} + \frac{1}{\delta^2}h + \frac{1}{2}h'' = 0, & \text{on } \Omega_2(h) \\ \frac{\partial h}{\partial t} + \frac{1}{\delta^2}h + \frac{1}{4}h'' + sign(h'')\frac{1}{2\delta}|h'| & \\ \quad + \frac{1}{4\delta^2}\frac{(h')^2}{h''} = 0, & \text{on } \Omega_3(h) \\ h(x, 0) = S(x) & \forall x \in \Omega. \end{cases} \quad (15)$$

Certainly, we need to add a boundary condition on $\partial\Omega$, which will depend on the input signal S. For instance, if $S = 0$ on $\partial\Omega$, we shall take a Dirichlet boundary condition. On the other hand, if $S' = 0$ on $\partial\Omega$, a Neumann boundary condition or more generally, a mixed Robbin type boundary condition is considered. Next, we derive from the PDE (15), a particular model by only considering the second equation assumed to be well posed everywhere.

Definition 3: A $Heat\ p\ \delta - IMF$ $(H\ p\ \delta - IMF)$ is a function k defined as

$$\begin{cases} k: & \Omega \longrightarrow \mathbb{R} \\ & x \longmapsto k(x) = h(x, t_0) \end{cases} \quad (16)$$

for some $t_0 > 0$, where h is solution of the PDE:

$$\begin{cases} \frac{\partial h}{\partial t} + \frac{1}{\delta^2}h + \frac{1}{2}h'' = 0, \\ h(x, 0) = S(x)\ \forall x \in \Omega \end{cases}. \quad (17)$$

We need of course to specify boundary conditions: we shall consider mainly Dirichlet or Neumann type conditions. The PDE (17) could be rewritten as a backward Heat equation, and is shown to be well posed in [8]. Concepts of δ-IMF are introduced as follows.

Definition 4: We say that a function h is a δ-IMF if and only if

1) h is a solution of the PDE (17);
2) $h(., T)$ is a null mean function for some T.

In practice, a δ-IMF is extracted when its mean is almost zero. Once the first δ-IMF, $\delta - IMF_1$, is extracted by resolving (17), we resolve again (17) to get $\delta - IMF_2$, but the initial condition is now equal to the residual between the signal and $\delta - IMF_1$; and so on for other δ-IMFs.

Definition 6: δ-EMD is the decomposition for which δ-IMFs are extracted with (17).

III. Numerical Results

PDEs are implemented with an explicit scheme. We first consider the following signal:

$$S = S_1 + S_2, \text{ where } S_1(x) = \cos\left(\frac{2}{\pi}x\right) \text{ and}$$
$$S_2(x) = \cos\left(\frac{2}{\pi}fx\right). \tag{18}$$

The two first modes are displayed in Figs. 1(a) and (b), respectively. A first remark is that the δ-EMD behaves like the classical EMD, in the sense that it separates the signal's components from the highest frequency to the lowest one. One could notice the small attenuation of the δ-IMFs as regards to the exact values of the signal and the IMFs. δ-IMFs are obtained with Neumann boundary conditions, $\delta = 0.3486$ and $T = 0.6750$. The second simulated signal is given by

$$S(x) = S_1(x) + S_2(x), \text{ where } S_1(x) = 2\sin\left(\frac{20}{\pi}x\right) \text{ and}$$
$$S_2(x) = 3\sin\left(\frac{2}{\pi}x\right). \tag{19}$$

Figs. 2(a) and (b) show the first two modes, which should normally equal to S_1 and S_2 respectively. The signal's components are very well separated by our approach. $\delta-\mathrm{IMF}_1$ and $\delta-\mathrm{IMF}_2$ fit exactly S_1 and S_2 respectively. On the other hand, IMF_1 is almost the same as S_1, except at boundaries where we notice some little differences (Fig. 2(a)). On the contrary, IMF_2 totally differs from S_2 (Fig. 2(b)). Probable reasons for that are the well known boundary problems that occur during the SP, and are obviously due to the mean envelopes computed by interpolations (splines for examples). The last signal is

$$S = S_1 + S_2, \text{ where}$$
$$S_1(x) = 4\sin(20\pi x)\sin(0.2\pi x) \text{ and}$$
$$S_2(x) = \sin(10\pi x). \tag{20}$$

Despite a little attenuation of the amplitude of $\delta-\mathrm{IMF}_2$ (Fig. 3(b)), this example clearly illustrates again the relevance and efficiency of the proposed δ-EMD. For the last two examples, δ-IMFs are obtained with Dirichlet boundary conditions, $\delta = 0.3054$ and $T = 0.7461$, and $\delta = 0.0636$ and $T = 0.8100$, respectively.

Fig. 4 shows comparisons between δ-IMFs and modes obtained by implementing (15), for the first (cf. (20)) and third signal [cf. (20)] considered before [Figs. 4(a)–(d)]. We assume that $h(., T)$ is a null mean function for some T, with h as solution of (15). There is no attenuation and modes fit exactely the first signal's components [Fig. 1(a)–(b) versus Fig. 4(a)–(b)]. In fact, PDE (15) takes into account critical points of the signal and points that have a null second derivative. Also, it accounts the concavity and the convexity of the signal, and handles points for which the signal's first derivative is above or beneath the second derivative.

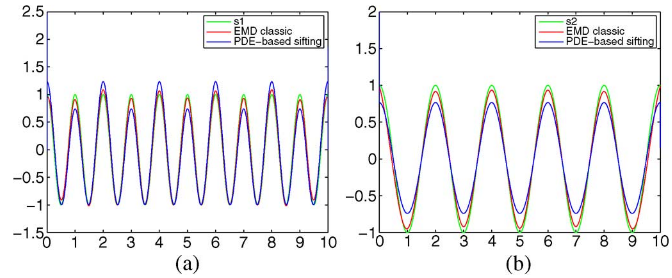

Fig. 1. Comparisons between classical IMFs and δ-IMFs for the first signal given by (18).

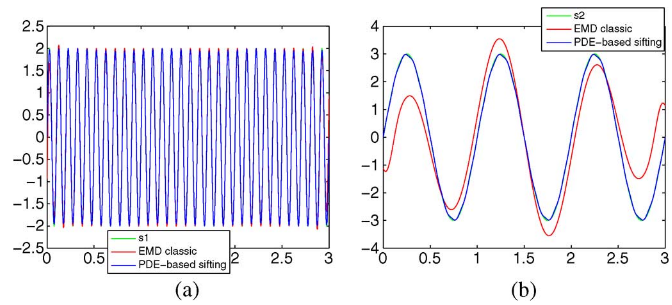

Fig. 2. Comparisons between classical IMFs and δ-IMFs for the second signal given by (19).

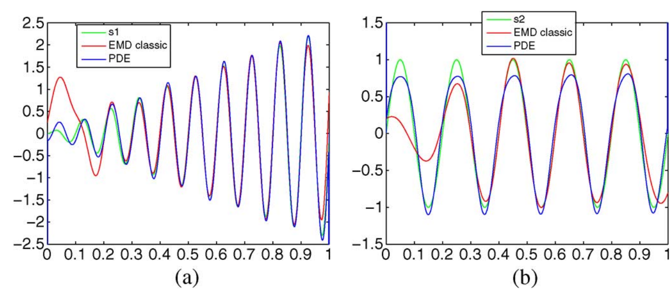

Fig. 3. Comparisons between classical IMFs and δ-IMFs for the last signal given by (20).

IV. Conclusion

Some theoretical contributions on the comprehension and the formalism of the EMD are provided here. In fact, we propose to perform the SP by the resolution of PDEs, and analytical characterizations of modes are then proposed. Moreover, a suitable way for getting rid of interpolation's issues is also provided. Further investigations of the PDE based approach should be done. Indeed, the parameter δ is actually chosen empirically, and we make the assumption that $O(\delta)\|h'''\|_\infty = 0$. The stopping criterion needs also some refinements, because if the signal has a null mean, then the algorithm stops at the first iteration, which means that any function that has a null mean is then a δ-IMF. An alternative to that is to consider the free-boundary PDE (15).

Appendix

Before giving the proof of Theorem 1, we first introduce operators S_δ and I_δ, respectively defined for all $h \in L^\infty(\Omega)$ by $S_\delta h(x) = \sup_{|y|<\delta} h(x+y)$ and $I_\delta h(x) = \inf_{|y|<\delta} h(x+y)$,

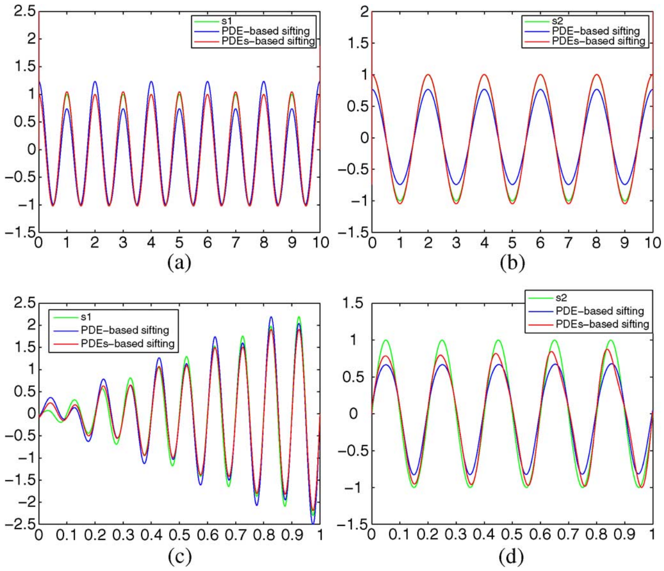

Fig. 4. Comparisons between δ-IMFs and obtained modes with (15). (a), (b): first signal (cf. (19)). (c), (d): third signal (cf. (19)). In green: signal's components, in blue: $\delta - IMFs$ and in red: modes obtained with (15).

$\forall x \in \Omega$ [8]. Then, m_δ (cf. Definition 2) is simply written as $m_\delta h(x) = (1/2)[S_\delta h(x) + I_\delta h(x)]$.

Theorem 2: Let $h \in C^3(\bar{\Omega})$ so that h''' is bounded on Ω. Then for $\delta > 0$ small, one has

$$S_\delta h(x) = h(x) + \sup_{|z|\le 1} F_{x,\delta}(z) + O(\delta^3)\|h'''\|_\infty,\ \forall x \in \Omega \quad (21)$$

where $F_{x,\delta}(z) = \delta z h'(x) + (\delta^2/2)z^2 h''(x)$. Furthermore

$$\sup_{|z|\le 1} F_{x,\delta}(z) = \begin{cases} -\frac{1}{2}\frac{(h'(x))^2}{h''(x)} & \text{if } |h'(x)| \le \delta\,|h''(x)| \text{ and } h''(x) < 0 \\ \delta\,|h'(x)| + \frac{\delta^2}{2}h''(x) & \text{otherwise.} \end{cases} \quad (22)$$

Proof: For any $x \in \Omega$, for any $z \in \mathbb{R}$ such that $|z| < 1$ and $x + \delta z \in \Omega$, since $h \in C^3(\bar{\Omega})$, we can write, using Taylor's formula: $h(x + \delta z) = h(x) + \delta z h'(x) + (\delta^2/2)z^2 h''(x) + O(\delta^3)\|h'''\|_\infty$. By taking the supremum in z, in view of the Definition of S_δ, one gets the result, with the definition of F_δ. In order to prove the explicit formula for $\sup_{|z|\le 1} F_{x,\delta}(z)$ for any fixed $x \in \Omega$, we first note that: $F_{x,\delta}(1) = \delta h'(x) + (\delta^2/2)h''(x)$ and $F_{x,\delta}(-1) = -\delta h'(x) + (\delta^2/2)h''(x)$.

- **case 1**: If $h''(x) = 0$. Then $F_{x,\delta}(z) = \delta h'(x) z$ and thus

$$\sup_{|z|\le 1} F_{x,\delta}(z) = \delta\,|h'(x)|\,. \quad (23)$$

- **case 2**: If $h''(x) \neq 0$.
If $F'_{x,\delta}$ denotes the derivative of $F_{x,\delta}$ w.r.t z and so on, then: $F'_{x,\delta}(z) = \delta h'(x) + \delta^2 z h''(x)$, $F''_{x,\delta}(z) = \delta^2 h''(x)$. Note that in particular of course $F_{x,\delta}(.)$ is either convex or concave w.r.t. $z \in [-1, 1]$. Therefore $F'_{x,\delta}(z) = 0 \Leftrightarrow z = z_0 = -(1/\delta)(h'(x)/h''(x))$, and then $F_{x,\delta}(z_0) = -(1/2)((h'(x))^2/h''(x))$.
— If $|z_0| > 1$, that is $|h'(x)| > \delta|h''(x)|$, then:

$$\sup_{|z|\le 1} F_{x,\delta}(z) = \max\{F_{x,\delta}(-1), F_{x,\delta}(1)\} = \delta\,|h'(x)| + \frac{\delta^2}{2}h''(x). \quad (24)$$

— If $|z_0| \le 1$, that is $|h'(x)| \le \delta|h''(x)|$, then
  - If $h''(x) > 0$. This implies that $F_{x,\delta}$ is convex; and thus again

$$\sup_{|z|\le 1} F_{x,\delta}(z) = \delta\,|h'(x)| + \frac{\delta^2}{2}h''(x). \quad (25)$$

  - If $h''(x) < 0$, $F_{x,\delta}$ is then concave. Thus

$$\sup_{|z|\le 1} F_{x,\delta}(z) = F_\delta(z_0) = -\frac{1}{2}\frac{(h'(x))^2}{h''(x)}. \quad (26)$$

A similar result for I_δ is of course true.

Theorem 3: Let $h \in C^3(\bar{\Omega})$ so that h''' is bounded on Ω. For $\delta > 0$ small, one has

$$I_\delta h(x) = h(x) + \inf_{|z|\le 1} F_{x,\delta}(z) + O(\delta^3)\|h'''\|_\infty,\ \forall x \in \Omega \quad (27)$$

where again $F_{x,\delta}(z) = \delta z h'(x) + (\delta^2/2)z^2 h''(x)$. Furthermore

$$\inf_{|z|\le 1} F_{x,\delta}(z) = \begin{cases} -\frac{1}{2}\frac{(h'(x))^2}{h''(x)} & \text{if } |h'(x)| \le \delta\,|h''(x)| \text{ and } h''(x) > 0 \\ -\delta\,|h'(x)| + \frac{\delta^2}{2}h''(x) & \text{otherwise.} \end{cases} \quad (28)$$

Proof of Theorem 1: Just consider results obtained for S_δ and I_δ in Theorems 2 and 3, respectively.

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