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# Separated representation of incremental elastoplastic simulations

Mohamed Aziz Nasri<sup>1,a</sup>, Jose Vicente Aguado<sup>2,b</sup>, Amine Ammar<sup>1,c</sup>,  
Elias Cueto<sup>3,d</sup>, Francisco Chinesta<sup>2,e</sup>, Franck Morel<sup>1,f</sup>, Camille Robert<sup>2,g</sup>  
and Saber Elareem<sup>1,h</sup>

<sup>1</sup>LAMPA, ENSAM Angers, France

<sup>2</sup>GeM Institute, Ecole Centrale de Nantes, Nantes, France

<sup>3</sup>I3A, Univzersity of Zaragoza, Spain

<sup>a</sup>Mohamed-Aziz.NASRI@ensam.eu, <sup>b</sup>jose.aguado-lopez@ec-nantes.fr,

<sup>c</sup>Amine.AMMAR@ensam.eu, <sup>d</sup>ecueto@unizar.es, <sup>e</sup>Francisco.Chinesta@ec-nantes.fr,

<sup>f</sup>Franck.Morel@ensam.eu, <sup>g</sup>Camille.ROBERT@ensam.eu, <sup>h</sup>Saber.ELAREM@ensam.eu

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**Abstract.** Forming processes usually involve irreversible plastic transformations. The calculation in that case becomes cumbersome when large parts and processes are considered. Recently Model Order Reduction techniques opened new perspectives for an accurate and fast simulation of mechanical systems, however nonlinear history-dependent behaviors remain still today challenging scenarios for the application of these techniques. In this work we are proposing a quite simple non intrusive strategy able to address such behaviors by coupling a separated representation with a POD-based reduced basis within an incremental elastoplastic formulation.

## Introduction

The main difference between viscoplasticity and elastoplasticity is that in elastoplasticity the behavior at each position and time depends on all the previous mechanical history as well as on the present loading. Obviously if the deformation history is given at each position, the elastoplastic behavior law can be easily integrated in order to compute the stress evolution. However such an information is not generally available when proceeding with standard discretization strategies where the solution is computed incrementally.

When using standard incremental discretization techniques we must proceed differently. We assume at time  $t_n$  (time in the sense of loading) the state of the systems perfectly defined, verifying the equilibrium and the constitutive equations. The mechanical state is given by  $\epsilon_n$ ,  $\sigma_n$ ,  $\epsilon_n^p$  and  $\alpha_n$ , the last being the internal variable describing the material hardening (accumulated plastic deformation, plastic work, ...) assumed isotropic without loss of generality.

The equilibrium at time  $t_n$  writes in discrete form:

$$\mathbf{F}^{int}(\sigma_n) = \mathbf{F}_n^{ext} \quad (1)$$

Now, an increment of load applies due to volume forces or surface tractions, and the problem consists in computing the displacement change  $\Delta u_{n+1}$  and the associated final state defined by  $\epsilon_{n+1}$ ,  $\sigma_{n+1}$ ,  $\epsilon_{n+1}^p$  and  $\alpha_{n+1}$ . The new state must verify both the equilibrium

$$\mathbf{F}^{int}(\sigma_{n+1}) = \mathbf{F}_{n+1}^{ext} \quad (2)$$

and the rate-independent elastoplastic constitutive equations. As the problem is nonlinear we must iterate. We denote by  $\Delta \mathbf{U}_{n+1}^k$  the nodal displacements increment at the loading step  $t_n$  and the nonlinear iteration  $k$ .

As soon as the nodal displacement increment  $\Delta \mathbf{U}_{n+1}^k$  is given, we can compute  $\sigma_{n+1}^k$  by using an appropriate return mapping [10] that allows after assembling evaluating the internal contributions  $\mathbf{F}^{int}(\sigma_{n+1}^k)$ .

The displacement increment results from the Newton linearization of Eq. (2) that results in

$$\mathbf{F}^{int}(\sigma_{n+1}^k) - \mathbf{F}_{n+1}^{ext} = \mathbf{K}_{n+1}^k \Delta \mathbf{U}_{n+1}^{k+1} \quad (3)$$

where in the calculation of the tangent matrix  $\mathbf{K}_{n+1}^k$  the return mapping algorithm is considered again. The possibility of applying a space separated representation for alleviating the complexity of 3D solutions is being evaluated.

The main difficulty associated with such an approach is its incremental character that makes difficult the use of model order reduction techniques based on the use of separated representations.

The LATIN method is one exception. The LATIN method, proposed by Pierre Ladeveze in the 80s [8] proceeds by assuming a space-time representation of both the deformation and the stress fields. It consists of two stages. In the first, stresses and strains are calculated in order to verify the constitutive equation, by assuming a certain (a priori arbitrary) relation between both fields, that is: knowing at iteration  $k$  (iteration in the nonlinear sense)  $\epsilon^k(\mathbf{x}, t)$  and  $\sigma^k(\mathbf{x}, t)$ , we look for  $\hat{\epsilon}^k(\mathbf{x}, t)$  and  $\hat{\sigma}^k(\mathbf{x}, t)$ , such that

$$\begin{cases} \hat{\epsilon}^k(\mathbf{x}, t) = \mathcal{F}(\hat{\sigma}^k(\mathbf{x}, t)) \\ \hat{\epsilon}^k(\mathbf{x}, t) - \epsilon^k(\mathbf{x}, t) = \mathcal{K}(\hat{\sigma}^k(\mathbf{x}, t) - \sigma^k(\mathbf{x}, t)) \end{cases} \quad (4)$$

where  $\mathcal{F}()$  denotes the behavior law and  $\mathcal{K}$  the searching direction. This calculation is purely local. The computed fields verify the constitutive equation, however they do not verify the equilibrium.

Then, a new stress-strain couple  $\epsilon^{k+1}(\mathbf{x}, t)$  and  $\sigma^{k+1}(\mathbf{x}, t)$  is searched in order to fulfill with the equilibrium equation and a particular (a priori arbitrary) searching direction:

$$\begin{cases} \int_{\Omega \times \mathcal{I}} \epsilon^* : \sigma^{k+1} d\mathbf{x} dt = \int_{\partial\Omega \times \mathcal{I}} u^* \cdot \mathbf{t} d\mathbf{x} dt \\ \sigma^{k+1}(\mathbf{x}, t) - \hat{\sigma}^{k+1}(\mathbf{x}, t) = \mathcal{G}(\epsilon^{k+1}(\mathbf{x}, t) - \hat{\epsilon}^{k+1}(\mathbf{x}, t)) \end{cases} \quad (5)$$

where  $\mathbf{t}$  denotes the tractions,  $\mathcal{I}$  the time interval and  $\mathcal{G}$  the new searching direction. Second relation in Eq. (5) ensures the existence of second derivatives of the displacement field in the equilibrium equation. The trickiest issue of this approach is related to the choice of the searching directions  $\mathcal{K}$  and  $\mathcal{G}$ .

In the present work we consider an integration that avoids the use of such searching direction and at the same time allows for the consideration of space and time separated representations all within an incremental explicit elastoplastic formulation.

### Incremental elastoplastic model

The elastic behavior is given by

$$\sigma = \mathbf{C} \epsilon \quad (6)$$

The main steps of the calculation are:

- Compute the deviatoric stress  $\sigma'$  from

$$\sigma' = \sigma - (1 + \nu) Tr(\sigma) \quad (7)$$

where  $\nu$  is the Poisson coefficient.

- Determine the equivalent stress

$$\sigma_e = \sqrt{\frac{3}{2} \sigma' : \sigma'} \quad (8)$$

- Determine the yield function

$$f = \sigma_e - (r + \sigma_y) \quad (9)$$

where  $r$  is the hardening and  $\sigma_y$  the yield stress.

- Determine if yielding occurs, that is, if  $f > 0$ . If  $f \leq 0$  then the plastic multiplier  $d\lambda$  vanishes  $d\lambda = 0$ . On the contrary, i.e. if  $f > 0$  the plastic multiplier results from

$$d\lambda = \frac{\mathbf{n} \cdot \mathbf{C} d\boldsymbol{\epsilon}}{\mathbf{n} \cdot \mathbf{C} \mathbf{n} + h} \quad (10)$$

with  $h$  the plastic tangent modulus and  $\mathbf{n}$  given by

$$\mathbf{n} = \frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{3}{2} \frac{\boldsymbol{\sigma}'}{\sigma_e} \quad (11)$$

- Determine the stress and the isotropic hardening increments:

$$d\boldsymbol{\sigma} = \mathbf{C} d\boldsymbol{\epsilon}^e = \mathbf{C} (d\boldsymbol{\epsilon} - d\lambda \mathbf{n}) \quad (12)$$

and

$$dr = h d\lambda \quad (13)$$

- Update all the quantities

$$\begin{cases} \sigma(\mathbf{x}, t + \Delta t) = \sigma(\mathbf{x}, t) + d\sigma \\ \boldsymbol{\epsilon}^p(\mathbf{x}, t + \Delta t) = \boldsymbol{\epsilon}^p(\mathbf{x}, t) + d\lambda \mathbf{n} \\ r(t + \Delta t) = r(t) + dr \end{cases} \quad (14)$$

The equilibrium writes (using vector notation)

$$\int_{\Omega} \boldsymbol{\epsilon}^* \cdot d\boldsymbol{\sigma} \, d\mathbf{x} = \int_{\partial\Omega} \mathbf{u}^* \cdot d\boldsymbol{\sigma} \, d\mathbf{x} \quad (15)$$

or

$$\int_{\Omega} \boldsymbol{\epsilon}^* \cdot \mathbf{C} d\boldsymbol{\epsilon} \, d\mathbf{x} = \int_{\Omega} \boldsymbol{\epsilon}^* \cdot \mathbf{C} d\lambda \mathbf{n} \, d\mathbf{x} + \int_{\partial\Omega} \mathbf{u}^* \cdot d\boldsymbol{\sigma} \, d\mathbf{x} \quad (16)$$

Now, a standard finite element discretization results in

$$\int_{\Omega} \boldsymbol{\epsilon}^* \cdot \mathbf{C} d\boldsymbol{\epsilon} \, d\mathbf{x} = \mathbf{U}^{*T} \mathbf{K} d\mathbf{U} \quad (17)$$

and

$$\int_{\Omega} \boldsymbol{\epsilon}^* \cdot \mathbf{C} d\lambda \mathbf{n} \, d\mathbf{x} + \int_{\partial\Omega} \mathbf{u}^* \cdot d\boldsymbol{\sigma} \, d\mathbf{x} = \mathbf{U}^{*T} \mathbf{F} \quad (18)$$

## Space-time separated representation

If we consider  $n$  time steps (again time in the sense of loading) we can write

$$\begin{cases} \mathbf{KdU}_1 = \mathbf{F}_1 \\ \mathbf{KdU}_2 = \mathbf{F}_2 \\ \vdots \\ \mathbf{KdU}_n = \mathbf{F}_n \end{cases} \quad (19)$$

Now, we can apply the singular value decomposition to the matrix  $\mathcal{F}$

$$\mathcal{F} = [\mathbf{F}_1 \ \mathbf{F}_2 \ \cdots \ \mathbf{F}_n] \quad (20)$$

that allows approximating it from

$$\mathcal{F} \approx \sum_{i=1}^{N_F} \mathbf{R}_i \otimes \mathbf{S}_i \quad (21)$$

With the unknown field  $\mathbf{dU}(t)$  expressed in the separated non-incremental form

$$\mathbf{dU} = \sum_{j=1}^{j=N_U} (\mathbf{X}_j \otimes \mathbf{T}_j) \quad (22)$$

and using tensor notation the problem becomes

$$(\mathbf{K} + \mathbf{I}) \sum_{j=1}^{j=N_U} (\mathbf{X}_j \otimes \mathbf{T}_j) = \sum_{i=1}^{N_F} (\mathbf{R}_i \otimes \mathbf{S}_i) \quad (23)$$

For more details on the use of separated representations the interested reader can refer to [1] [2] [3] [4] [5] [7] and the references therein.

The main drawback of such a procedure is the necessity of reconstructing the plastic history and then applying a SVD to the large resulting matrix to perform its space-time separation. It is important to recall that the number of snapshots  $n$  in (20) could be extremely large. To alleviate such a calculation in the next section we propose the use of a POD-based reduced basis for the plastic history representation.

## POD-based reduced modeling

To avoid the application of the SVD on the whole matrix  $\mathcal{F}$  defined in (20) we consider few number of snapshots  $p$ , with  $p$  large enough but  $p \ll n$ , and define matrix  $\mathcal{Q}_1$

$$\mathcal{Q}_1 = [\mathbf{F}_1 \ \mathbf{F}_2 \ \cdots \ \mathbf{F}_p] \quad (24)$$

then  $\mathcal{Q}_2$  from

$$\mathcal{Q}_2 = [\mathbf{F}_{p+1} \ \mathbf{F}_2 \ \cdots \ \mathbf{F}_{2p}] \quad (25)$$

and so on.

Now the SVD can be applied on the different matrix  $\mathcal{Q}_j$  for extracting the main important modes of each group

$$\mathcal{B}_j = [\phi_1^j \ \phi_2^j \ \cdots \ \phi_{r_j}^j] \quad (26)$$

By projecting the snapshots of each group  $\mathcal{Q}_j$  into their respective reduced basis  $\mathcal{B}_j$  we can write

$$\mathcal{Q}_j = \mathcal{B}_j \cdot \mathcal{A}_j \quad (27)$$

that allows writing

$$\mathcal{B} = [\mathcal{B}_1 \mathcal{B}_2 \cdots] \quad (28)$$

and  $\mathcal{A}$  whose blocs components are  $\mathcal{A}_j$  such that

$$\mathcal{F} \approx \mathcal{B} \cdot \mathcal{A} \quad (29)$$

The searched separated representation of the term representing the plastic history

$$\mathcal{F} \approx \sum_{i=1}^{N_F} \mathbf{R}_i \otimes \mathbf{S}_i \quad (30)$$

involves modes  $\mathbf{R}_i$  coming from the SVD decomposition of  $\mathcal{B}$  and modes  $\mathbf{S}_i$  result the projection of  $\mathcal{A}$  into the the basis now involving modes  $\mathbf{R}_i$ .

Even if many other strategies exist, as for example a progressive construction and enrichment of the reduced basis [9], the one described above allows a significant computing time reduction.

### Numerical results

We apply on a rectangular specimen two kind of uniaxial cyclic displacements on its opposite borders, one with constant mean value and the other with a mean value that increases linearly with the time, both illustrated in Fig. 1.

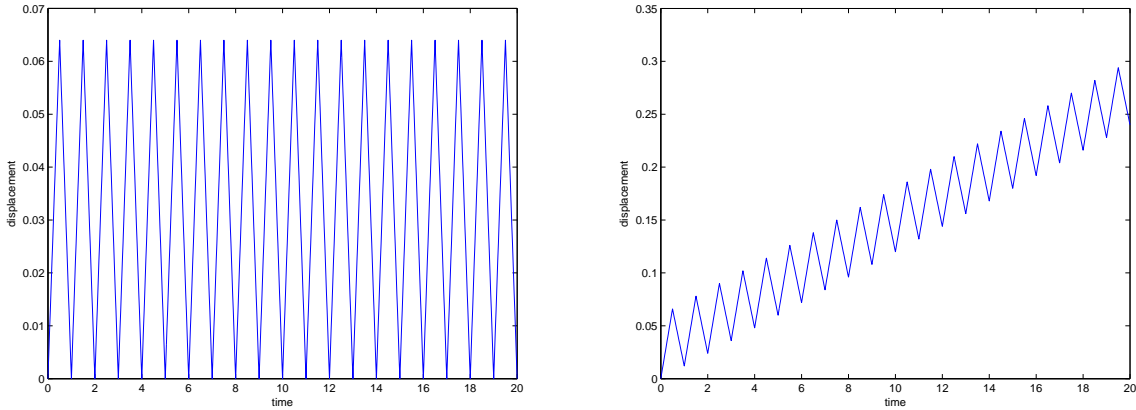


Fig. 1: Applied cyclic displacements: (left) constant mean value; (right) displacement mean value increasing in time

Fig. 2 depicts the stress  $\sigma_{xx}$  evolution in both cases (constant and increasing displacement mean value) at the center of the specimen, both solutions computed without any model order reduction, that is, by applying a standard incremental explicit elastoplastic integration.

Fig. 4 compares the first three more significant space modes involved in the separated representation of the displacement increment field, that is  $\mathbf{X}_i$ ,  $i = 1, 2, 3$ , for both the constant and the increasing cyclic loadings. Finally from the reconstructed solution at the central point after each solution enrichment, Fig. 4 depicts the whole history of the stress versus the applied displacement. It can be noticed that the reduced solutions converge in both loading cases to the reference ones (the ones depicted in Fig. 2) after only 3 enrichment steps.

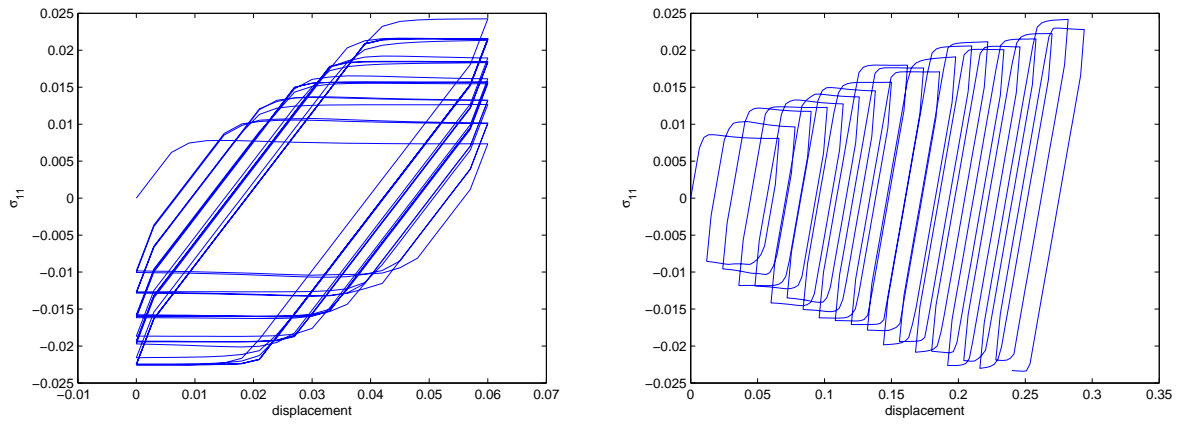


Fig. 2: Stress versus applied displacement at the center of the specimens for both (left) constant and (right) increasing mean value of the applied displacement.

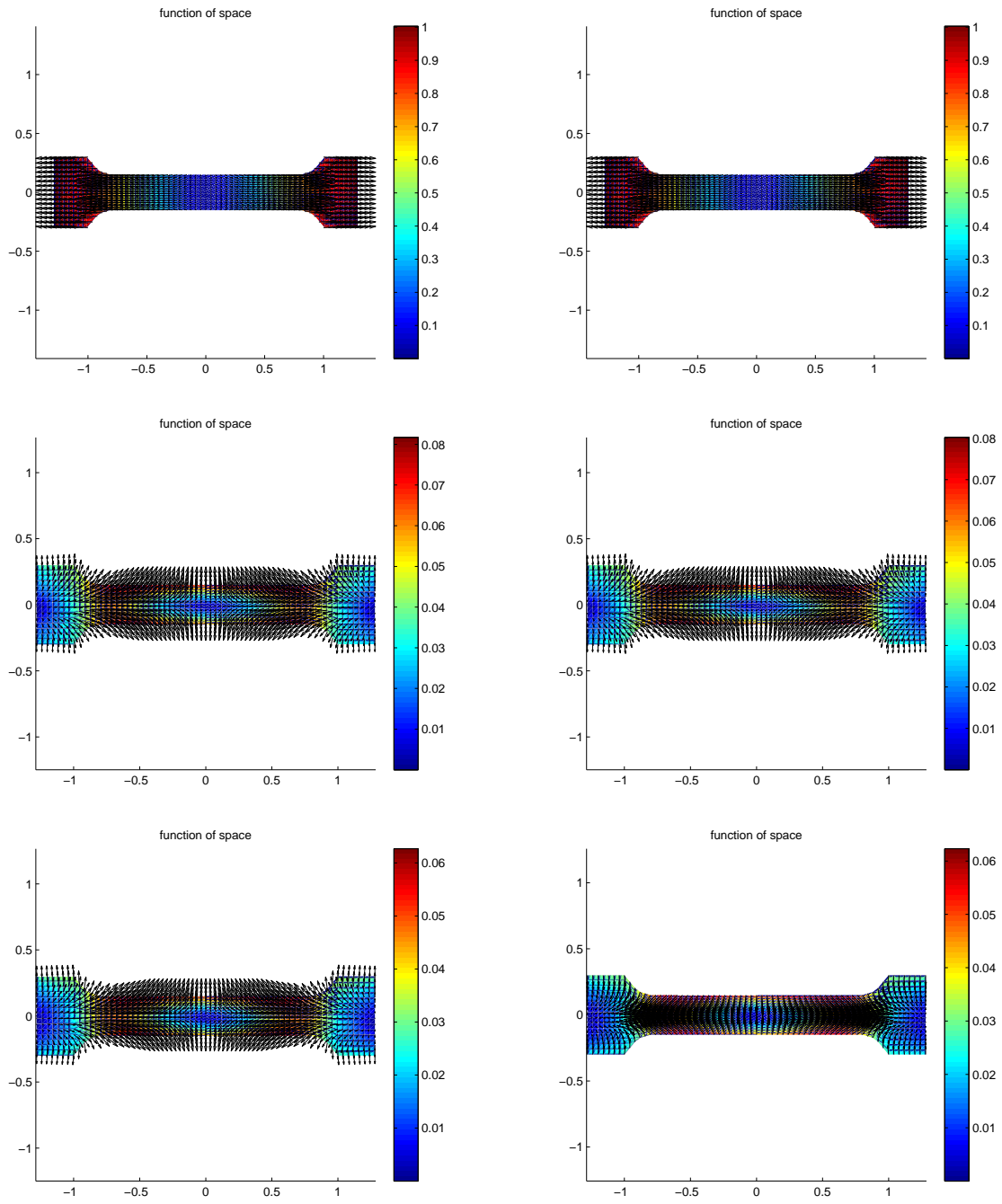


Fig. 3: First three space modes involved in the separated representation of the displacement increment  $dU$ : (left) constant and (right) increasing mean displacement



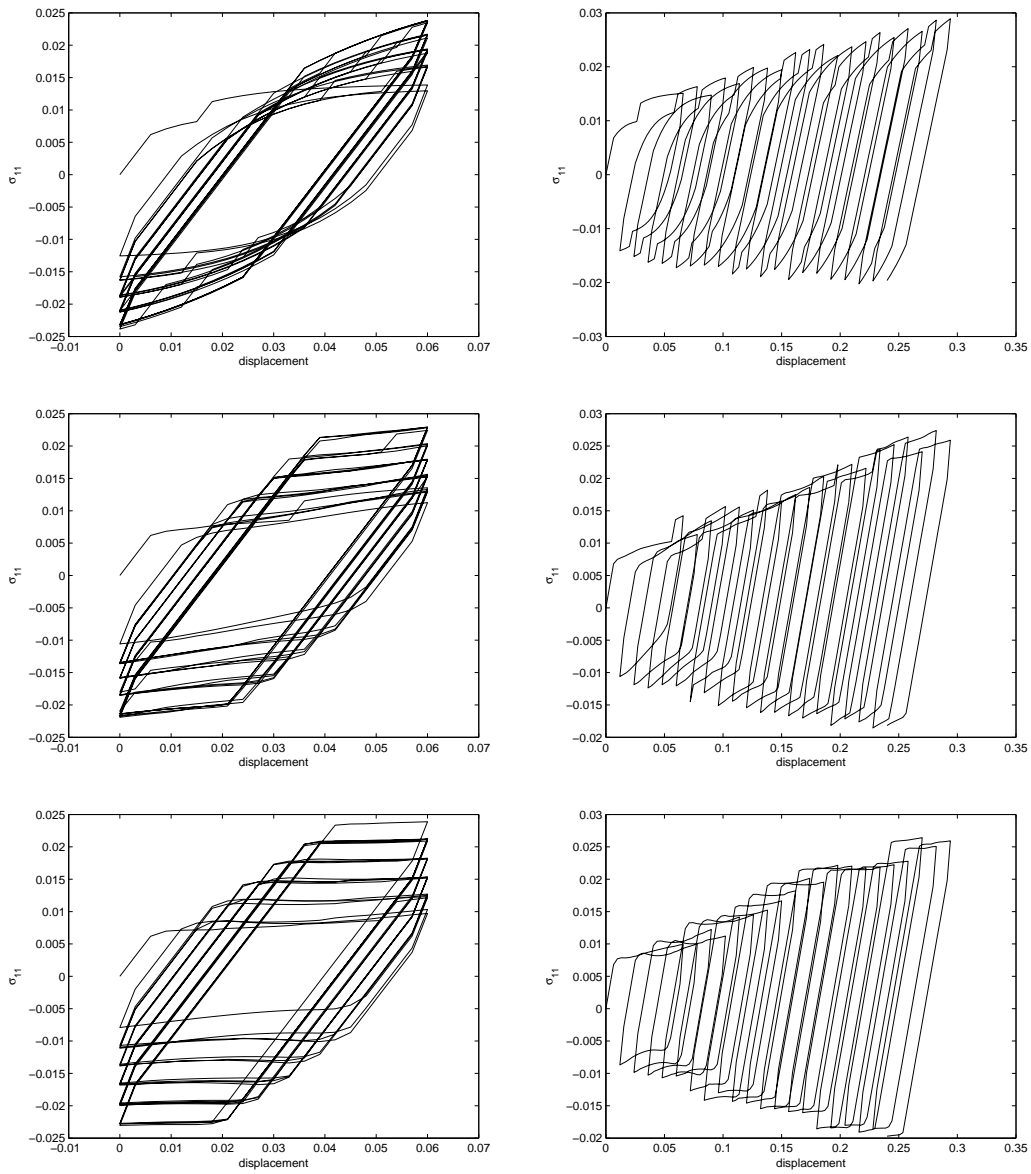


Fig. 4: Stress versus applied displacement at the center of the specimen for one, two and three terms involved in the separated representation (from top to down) and (left) constant and (right) increasing mean displacement

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