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An extension of the Polar Method to the First-order Shear Deformation Theory of laminates

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Abstract

In this paper the Verchery's polar method is extended to the conceptual framework of the First-order Shear Deformation Theory (FSDT) of laminates. It will be proved that the number of independent tensor invariants characterising the laminate constitutive behaviour remains unchanged when passing from the context of the Classical Laminate Theory (CLT) to that of the FSDT. Moreover, it will also be shown that, depending on the considered formulation, the elastic symmetries of the laminate shear stiffness matrix depend upon those of membrane and bending stiffness matrices. As a consequence of these results a unified formulation for the problem of designing the laminate elastic symmetries in the context of the FSDT is proposed. The optimum solutions are found within the framework of the polar-genetic approach, since the objective function is written in terms of the laminate polar parameters, while a genetic algorithm is used as a numerical tool for the solution search. In order to support the theoretical results, and also to prove the effectiveness of the proposed approach, some novel and meaningful numerical examples are discussed in the paper.

Keywords:

Anisotropy; Polar method; Genetic Algorithms; Composite materials; Structural design.

Notations

CLT, Classical Laminate Theory

FSDT, First-order Shear Deformation Theory

GA, Genetic Algorithm

$\Gamma = \{O; x_1, x_2, x_3\}$, local (or material) frame of the elementary ply

$\Gamma^{\mathrm{I}} = \{O; x, y, z = x_3\}$, global frame of the laminate

θ, rotation angle

$\{11, 22, 33, 32, 31, 21\} \Leftrightarrow \{1, 2, 3, 4, 5, 6\}$, correspondence between tensor and Voigt's (matrix) notation for the indexes of tensors (local frame)

$\{xx, yy, zz, zy, zx, yx\} \Leftrightarrow \{x, y, z, q, r, s\}$, correspondence between tensor and Voigt's (matrix) notation for the indexes of tensors (global frame)

Z_{ij}, $(i, j = 1, 2$ or $i, j = x, y)$, second-rank plane tensor using tensor notation (local and global frame)

L_{ijkl}, $(i, j, k, l = 1, 2$ or $i, j, k, l = x, y)$, fourth-rank plane tensor using tensor notation (local and global frame)

U_i, $(i = 1, ..., 7)$ parameters of Tsai and Pagano

[Q], 3×3 in-plane reduced stiffness matrix of the constitutive lamina (Voigt's notation)

$[\widehat{\mathrm{Q}}]$, 2×2 out-of-plane reduced stiffness matrix of the constitutive lamina (Voigt's notation)

$T_0, T_1, R_0, R_1, \Phi_0, \Phi_1$, polar parameters of a fourth-rank plane tensor (also used for the lamina in-plane reduced stiffness matrix [Q])

T, R, Φ, polar parameters of a second-rank plane tensor (also used for the lamina out-of-plane reduced stiffness matrix $[\widehat{\mathrm{Q}}]$)

$\{\mathrm{N}\}$, 3×1 vector of membrane forces (per unit length), Voigt's notation

$\{\mathrm{M}\}$, 3×1 vector of bending moments (per unit length), Voigt's notation

$\{\mathrm{F}\}$, 2×1 vector of shear forces (per unit length), Voigt's notation

$\{\varepsilon_0\}$, 3×1 vector of in-plane strains of the laminate middle plane, Voigt's notation

$\{\chi_0\}$, 3×1 vector of curvatures of the laminate middle plane, Voigt's notation

$\{\gamma_0\}$, 2×1 vector of the out-of-plane shear strains of the laminate middle plane, Voigt's notation

$[\mathrm{A}], [\mathrm{B}], [\mathrm{D}]$, 3×3 matrices of laminate membrane, membrane/bending coupling and bending stiffness, respectively (Voigt's notation)

$[\mathrm{A}^*], [\mathrm{B}^*], [\mathrm{D}^*]$, 3×3 matrices of laminate homogenised membrane, membrane/bending coupling and bending stiffness, respectively (Voigt's notation)

$[\mathrm{H}]$, 2×2 matrix of laminate out-of-plane shear stiffness, (Voigt's notation)

$[\mathrm{H}^*]$, 2×2 matrix of laminate homogenised out-of-plane shear stiffness, (Voigt's notation)

$[\mathrm{C}^*]$, 3×3 laminate homogeneity matrix

$T_{0A^*}, T_{1A^*}, R_{0A^*}, R_{1A^*}, \Phi_{0A^*}, \Phi_{1A^*}$, polar parameters of $[\mathrm{A}^*]$

$T_{0B^*}, T_{1B^*}, R_{0B^*}, R_{1B^*}, \Phi_{0B^*}, \Phi_{1B^*}$, polar parameters of $[\mathrm{B}^*]$

$T_{0D^*}, T_{1D^*}, R_{0D^*}, R_{1D^*}, \Phi_{0D^*}, \Phi_{1D^*}$, polar parameters of $[\mathrm{D}^*]$

$T_{H^*}, R_{H^*}, \Phi_{H^*}$, polar parameters of $[\mathrm{H}^*]$

E_i, $(i = 1, 2, 3)$, Young's moduli of the constitutive lamina (material frame)

G_{ij}, $(i, j = 1, 2, 3)$, shear moduli of the constitutive lamina (material frame)

ν_{ij}, $(i, j = 1, 2, 3)$, Poisson's ratios of the constitutive lamina (material frame)

t_{ply}, thickness of the constitutive lamina

n, number of layers

$\{\delta_k\}$ $(k = 1, ..., n)$, vector of the layers orientation angles

h, overall thickness of the laminate

Ψ, overall objective function for the problem of designing the elastic symmetries of the laminate

{f}, 21×1 vector of partial objective functions

[W], 21×21 positive semi-definite diagonal weight matrix

$\widehat{R}_{0A^*}, \widehat{R}_{1A^*}, \widehat{\Phi}_{0A^*}, \widehat{\Phi}_{1A^*}$ imposed values for the polar parameters of matrix [A*]

$\widehat{R}_{0D^*}, \widehat{R}_{1D^*}, \widehat{\Phi}_{0D^*}, \widehat{\Phi}_{1D^*}$ imposed values for the polar parameters of matrix [D*]

N_{pop}, number of populations

N_{ind}, number of individuals

N_{gen}, maximum number of generations

p_{cross}, crossover probability

p_{mut}, mutation probability

1 Introduction

The problem of designing a composite structure is quite cumbersome and can be considered as a multi-scale design problem. The complexity of the design process is actually due to two intrinsic properties of composite materials, i.e. the heterogeneity and the anisotropy. Although the heterogeneity gets involved mainly at the micro-scale (i.e. the scale of constitutive "phases", namely fibres and matrix), conversely the anisotropy intervenes at both meso-scale (that of the constitutive lamina) and macro-scale (that of the laminate). It is well known that the material properties (and more generally the mechanical response) of an anisotropic continuum depend upon the direction. A consequence of anisotropy consists in the fact that the mechanical response of the material depends upon a considerable number of parameters (i.e. 21 for a general triclinic material, 13 for the monoclinic case, nine for the orthotropic one, five for the transverse isotropic case and two for an isotropic material).

Normally the Cartesian representation of tensors is employed to describe the behaviour of an anisotropic material in terms of Young's moduli, shear moduli, Poisson's ratios, Chentsov's ratios and mutual influence ratios, see [1]. While on one hand the Cartesian representation seems to be the "most natural" representation to describe the anisotropy, on

the other hand it shows a major drawback: the above material parameters depend upon the coordinate system chosen for characterising the mechanical response of the continuum. As a consequence, the anisotropy of the material is described by a set of parameters which are not (tensor) invariant quantities and that represent the response of the material only in a particular frame and not in a general one.

Several alternative analytical representations can be found in literature. Some of them rely on the use of tensor invariants which allow for describing the mechanical behaviour of an anisotropic continuum through intrinsic material quantities. Of course, such representations do not imply a reduction in the number of parameters needed to fully characterise the material behaviour. Nevertheless, since these intrinsic material quantities are tensor invariants on one hand they allow to describe the mechanical response of the material regardless to the considered reference frame and on the other hand they let to better highlight some physical aspects that cannot be easily caught when using the Cartesian representation.

In the framework of the design of composite materials several analytical representations of (plane) anisotropy were developed in the past and among them the most commonly employed is that introduced by Tsai and Pagano [2]. In the context of this approach they introduce seven parameters $U_i, (i = 1, ..., 7)$ which are expressed in terms of the six independent Cartesian components of a fourth-rank elasticity-like plane tensor (i.e. a tensor having both major and minor symmetries) written in the local frame $\Gamma = \{O; x_1, x_2, x_3\}$:

$$\begin{aligned}
U_1 &= \frac{3L_{1111} + 2L_{1122} + 3L_{2222} + 4L_{1212}}{8} \ , \\
U_2 &= \frac{L_{1111} - L_{2222}}{2} \ , \\
U_3 &= \frac{L_{1111} - 2L_{1122} + L_{2222} - 4L_{1212}}{8} \ , \\
U_4 &= \frac{L_{1111} + 6L_{1122} + L_{2222} - 4L_{1212}}{8} \ , \\
U_5 &= \frac{L_{1111} - 2L_{1122} + L_{2222} + 4L_{1212}}{8} \ , \\
U_6 &= \frac{L_{1112} + L_{1222}}{2} \ , \\
U_7 &= \frac{L_{1112} - L_{1222}}{2} \ .
\end{aligned} \tag{1}$$

The main drawbacks of this representation are basically three: firstly not all parameters U_i are tensor invariants, secondly they do not have a simple and immediate physical meaning and, finally, they are not all independent. Indeed, U_5 can be expressed in terms of U_1 and U_4 as:

$$U_5 = \frac{(U_1 - U_4)}{2} \ . \tag{2}$$

In 1979 Verchery [3] introduced the polar method for representing fourth-rank elasticity-like plane tensors. This representation has been enriched and deeply studied later by Vannucci and his co-workers [4–8]. The polar method relies upon a complex variable transformation by taking inspiration from a classical technique often employed in analytical mechanics, see for instance the works of Kolosov [9] and Green and Zerna [10]. As it will be briefly described in Sec. 2, the main advantages of the polar formalism are at least three: a) it is a representation of anisotropy which is based on tensor invariants, b) such invariants have an immediate physical meaning which is linked to the different (elastic) symmetries of the tensor and c) the change of reference frame can be expressed in a straightforward way.

Concerning the problem of the design of a composite structure, the polar method has been applied, up to now, only in the framework of the Classical Laminate Theory (CLT) for different real-life engineering applications, see [11–17]. Nevertheless, the results obtained by using the polar method in the context of the CLT are not sufficiently accurate for those applications involving moderately thick (or thick) composite parts. To overcome this difficulty, in this work the polar method is extended and applied (for the first time) for representing the classic laminate stiffness matrices in the framework of the First-order Shear Deformation Theory (FSDT). In particular, depending on the assumed mathematical formulation for the out-of-plane shear stiffness matrix of the laminate, the expressions of its polar parameters will be analytically derived. Accordingly, the unified formulation for the problem of designing the laminate elastic symmetries, initially introduced by Vannucci [18] in the context of the CLT, has been modified and extended to the case of the FSDT. This problem is formulated as an unconstrained minimisation problem in the space of the full set of the laminate polar parameters (membrane, bending, membrane/bending coupling and shear). Due to its particular nature (i.e. a high non-convex optimisation problem in the space of the layers orientation angles), the solution search process is performed by using the last version of the genetic algorithm (GA) BIANCA [11, 12, 19]. Finally, in order to numerically prove and support the major analytical results found in this work, some meaningful and non-conventional examples are presented.

The paper is organised as follows: Section 2 recalls the fundamentals of the polar formalism and the related advantages. In Section 3 the polar method is applied in the framework of the FSDT, by highlighting the major analytical results. Section 4 presents the mathematical formulation of the problem of designing the elastic symmetries of a laminate as an optimisation problem and the generalisation of this formulation when considering the laminate behaviour in the context of the FSDT. Section 5 shows some numerical results in order to prove the effectiveness of the polar formalism when it is applied to the FSDT. Finally Section 6 ends the paper with some concluding remarks.

2 Fundamentals of the Polar Method

In this section the main results of the Polar Method introduced by Verchery in 1979 [3] are briefly recalled. The polar method is substantially a mathematical technique that allows for expressing any n-rank plane tensor through a set of tensor invariants. As a consequence, such a representation can be applied not only to elasticity-like tensors but also to any other asymmetric plane tensor, see for instance [20]. Mainly inspired by the work of Green and Zerna [10], Verchery makes use of a (very classical) mathematical technique based upon a complex variable transformation in order to easily represent the affine transformation (in this case a rotation) of a plane tensor after a change of reference frame. For a deeper insight in the matter the reader is addressed to [4].

In the framework of the polar formalism a second-rank (symmetric) tensor Z_{ij}, $(i, j = 1, 2)$, within the local frame Γ, can be stated as:

$$\begin{array}{rcccl} Z_{11} & = & T & + & R\cos 2\Phi \ , \\ Z_{12} & = & & & R\sin 2\Phi \ , \\ Z_{22} & = & T & - & R\cos 2\Phi \ , \end{array} \tag{3}$$

where T is the isotropic modulus, R the deviatoric one and Φ the polar angle. From Eq. (3) it can be noticed that the three independent Cartesian components of a second-rank plane symmetric tensor are expressed in terms of three polar parameters: among them only two are tensor invariants, i.e. T and R, while the last one, namely the polar angle Φ, is needed to fix the reference frame. The converse relations (giving the polar parameters in terms of Cartesian components) are:

$$\begin{array}{rcl} T & = & \dfrac{Z_{11} + Z_{22}}{2} \quad , \\ Re^{\mathrm{i}2\Phi} & = & \dfrac{Z_{11} - Z_{22}}{2} + \mathrm{i}Z_{12} \quad , \end{array} \tag{4}$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. For a second-rank plane tensor the only possible symmetry is the isotropy which can be obtained when the deviatoric modulus of the tensor is null, i.e. $R = 0$. Moreover, as stated in the introduction, when using the polar formalism the components of the second-rank tensor can be expressed in a very straightforward manner in the frame Γ^{I} (turned counter-clock wise by an angle θ around the x_3 axis) as follows:

$$\begin{array}{rcccl} Z_{xx} & = & T & + & R\cos 2(\Phi - \theta) \ , \\ Z_{xy} & = & & & R\sin 2(\Phi - \theta) \ , \\ Z_{yy} & = & T & - & R\cos 2(\Phi - \theta) \ . \end{array} \tag{5}$$

Indeed the change of frame can be easily obtained by simply subtracting the angle θ from the polar angle Φ.

Concerning a fourth-rank elasticity-like plane tensor L_{ijkl}, $(i,j,k,l=1,2)$ (expressed within the local frame Γ), its polar representation writes:

$$
\begin{array}{lclclclclcl}
L_{1111} & = & & T_0 & + & 2T_1 & + & R_0\cos 4\Phi_0 & + & 4R_1\cos 2\Phi_1 & , \\
L_{1122} & = & - & T_0 & + & 2T_1 & - & R_0\cos 4\Phi_0 & & & , \\
L_{1112} & = & & & & & & R_0\sin 4\Phi_0 & + & 2R_1\sin 2\Phi_1 & , \\
L_{2222} & = & & T_0 & + & 2T_1 & + & R_0\cos 4\Phi_0 & - & 4R_1\cos 2\Phi_1 & , \\
L_{2212} & = & & & & & - & R_0\sin 4\Phi_0 & + & 2R_1\sin 2\Phi_1 & , \\
L_{1212} & = & & T_0 & & & - & R_0\cos 4\Phi_0 & & & .
\end{array}
\tag{6}
$$

As it clearly appears from Eq. (6) the six independent Cartesian components of L_{ijkl} are expressed in terms of six polar parameters: T_0 and T_1 are the isotropic moduli, R_0 and R_1 are the anisotropic ones, while Φ_0 and Φ_1 are the polar angles. Only five quantities are tensor invariants, namely the polar moduli T_0, T_1, R_0, R_1 together with the angular difference $\Phi_0-\Phi_1$. One of the two polar angles, Φ_0 or Φ_1, can be arbitrarily chosen to fix the reference frame. The converse relations can be stated as:

$$
\begin{array}{rcll}
8T_0 & = & L_{1111}-2L_{1122}+4L_{1212}+L_{2222} & , \\
8T_1 & = & L_{1111}+2L_{1122}+L_{2222} & , \\
8R_0e^{\mathrm{i}4\Phi_0} & = & L_{1111}-2L_{1122}-4L_{1212}+L_{2222}+4\mathrm{i}(L_{1112}-L_{2212}) & , \\
8R_1e^{\mathrm{i}2\Phi_1} & = & L_{1111}-L_{2222}+2\mathrm{i}(L_{1112}+L_{2212}) & .
\end{array}
\tag{7}
$$

Once again, thanks to the polar formalism it is very easy to express the Cartesian components of the fourth-rank tensor in the frame Γ^{I}, in fact it suffice to subtract the angle θ from the polar angles Φ_0 and Φ_1 as follows:

$$
\begin{array}{lclclclclcl}
L_{xxxx} & = & & T_0 & + & 2T_1 & + & R_0\cos 4(\Phi_0-\theta) & + & 4R_1\cos 2(\Phi_1-\theta) & , \\
L_{xxyy} & = & - & T_0 & + & 2T_1 & - & R_0\cos 4(\Phi_0-\theta) & & & , \\
L_{xxxy} & = & & & & & & R_0\sin 4(\Phi_0-\theta) & + & 2R_1\sin 2(\Phi_1-\theta) & , \\
L_{yyyy} & = & & T_0 & + & 2T_1 & + & R_0\cos 4(\Phi_0-\theta) & - & 4R_1\cos 2(\Phi_1-\theta) & , \\
L_{yyxy} & = & & & & & - & R_0\sin 4(\Phi_0-\theta) & + & 2R_1\sin 2(\Phi_1-\theta) & , \\
L_{xyxy} & = & & T_0 & & & - & R_0\cos 4(\Phi_0-\theta) & & & .
\end{array}
\tag{8}
$$

In the case of a fourth-rank elasticity-like tensor the real plus-value of the polar method is in the fact that the polar invariants are directly linked to the (elastic) symmetries of the tensor, thus having an immediate physical meaning. Indeed the polar formalism offers an *algebraic* characterization of the elastic symmetries, which can be seen as an alternative to the classical geometrical approach to the problem of finding the elastic symmetries of a material. In particular it can be proved that for a fourth-rank elasticity-like plane tensor four different types of elastic symmetry exist. They are briefly recalled in the following.

- *Ordinary orthotropy*: this symmetry corresponds to the algebraic condition

$$
\Phi_0-\Phi_1=K\frac{\pi}{4},\quad K=0,1\ . \tag{9}
$$

Indeed, for the same set of tensor invariants, i.e. T_0, T_1, R_0, R_1, two different shapes of orthotropy exist, depending on the value of K. Vannucci [4] shows that they correspond to the so-called *low* ($K = 0$) and *high* ($K = 1$) shear modulus orthotropic materials firstly studied by Pedersen [21]. However, this classification is rather limiting since the difference between these two classes of orhtotropy concerns, more generally, the global mechanical response of the material, see [4, 7].

- R_0-*Orthotropy*: the algebraic condition to attain this "special" orthotropy is

$$R_0 = 0 \ . \tag{10}$$

In this case the Cartesian components of the fourth-rank tensor L_{ijkl} change (as a result of a frame rotation) as those of a second-rank tensor, see Eqs. (3),(6). The existence of this particular orthotropy has been found also for the 3D case [22].

- *Square symmetry*: it can be obtained by imposing the following condition

$$R_1 = 0 \ . \tag{11}$$

This symmetry represents the 2D case of the well-known 3D cubic syngony.

- *Isotropy*: the fourth-rank elasticity-like tensor is isotropic when its anisotropic moduli are null, i.e. when the following condition is satisfied

$$R_0 = R_1 = 0 \ . \tag{12}$$

3 Application of the Polar Formalism to the First-order Shear Deformation Theory of laminates

For sake of simplicity in this section all of the equations governing the laminate mechanical response will be formulated in the context of the Voigt's (matrix) notation. The passage from tensor notation to Voigt's notation can be easily expressed by the following two-way relationships among indexes (for both local and global frames):

$$\begin{aligned} \{11, 22, 33, 32, 31, 21\} \quad &\Leftrightarrow \quad \{1, 2, 3, 4, 5, 6\} \quad , \\ \{xx, yy, zz, zy, zx, yx\} \quad &\Leftrightarrow \quad \{x, y, z, q, r, s\} \quad . \end{aligned} \tag{13}$$

Let us consider a multilayer plate composed of n identical layers (i.e. layers having same material properties and thickness). Let be δ_k the orientation angle of the k-th ply ($k = 1, ..., n$), t_{ply} the thickness of the elementary lamina and $h = nt_{\text{ply}}$ the overall thickness of the plate. In the framework of the FSDT theory [23] the constitutive law of the laminated plate (expressed within the global frame of the laminate Γ^{I}) can be stated as:

$$\left\{ \begin{array}{c} \{\text{N}\} \\ \{\text{M}\} \end{array} \right\} = \left[\begin{array}{cc} [\text{A}] & [\text{B}] \\ [\text{B}] & [\text{D}] \end{array} \right] \left\{ \begin{array}{c} \{\varepsilon_0\} \\ \{\chi_0\} \end{array} \right\} , \tag{14}$$

$$\{\mathrm{F}\} = [\mathrm{H}]\,\{\gamma_0\} \quad , \tag{15}$$

where [A], [B] and [D] are the membrane, membrane/bending coupling and bending stiffness matrices of the laminate, while [H] is the out-of-plane shear stiffness matrix. {N}, {M} and {F} are the vectors of membrane forces, bending moments and shear forces per unit length, respectively, whilst $\{\varepsilon_0\}$, $\{\chi_0\}$ and $\{\gamma_0\}$ are the vectors of in-plane strains, curvatures and out-of-plane shear strains of the laminate middle plane, respectively. The expressions of matrices [A], [B] and [D] are:

$$\begin{aligned} [\mathrm{A}] &= \frac{h}{n}\sum_{k=1}^{n}\left[\mathrm{Q}\left(\delta_k\right)\right] \quad , \\ [\mathrm{B}] &= \frac{1}{2}\left(\frac{h}{n}\right)^2\sum_{k=1}^{n} b_k\left[\mathrm{Q}\left(\delta_k\right)\right] \quad , \\ [\mathrm{D}] &= \frac{1}{12}\left(\frac{h}{n}\right)^3\sum_{k=1}^{n} d_k\left[\mathrm{Q}\left(\delta_k\right)\right] \quad , \end{aligned} \tag{16}$$

with

$$\begin{aligned} b_k &= 2k - n - 1 \quad , \quad \sum_{k=1}^{n} b_k = 0 \quad , \\ d_k &= 12k\left(k - n - 1\right) + 4 + 3n\left(n + 2\right) \quad , \quad \sum_{k=1}^{n} d_k = n^3 \quad . \end{aligned} \tag{17}$$

It can be noticed that in Eq. (16) $[\mathrm{Q}(\delta_k)]$ is the in-plane reduced stiffness matrix of the k-th ply. Concerning Eq. (15), in literature one can find different expressions for the out-of-plane shear stiffness matrix of the laminate [H]. In the following it will be considered two different representations for this matrix, namely:

$$[\mathrm{H}] = \begin{cases} \dfrac{h}{n}\displaystyle\sum_{k=1}^{n}[\widehat{\mathrm{Q}}(\delta_k)] & \text{(basic)} \quad , \\ \dfrac{5h}{12n^3}\displaystyle\sum_{k=1}^{n}(3n^2 - d_k)[\widehat{\mathrm{Q}}(\delta_k)] & \text{(modified)} \quad . \end{cases} \tag{18}$$

In Eq. (18) $[\widehat{\mathrm{Q}}(\delta_k)]$ is the out-of-plane shear stiffness matrix of the elementary ply. The first form of the matrix [H] is the basic one wherein the shear stresses are constant through the thickness of each lamina. However, as widely discussed in [1, 23] this approximation is not accurate at least for three reasons: a) a constant out-of-plane shear stress field does not satisfy the local equilibrium equations of each lamina, b) the shear stresses are discontinuous at the layers interfaces and c) the out-of-plane shear stresses must be null on both top and bottom surfaces of the laminated plate if no tangential forces are applied. To these purposes several modifications of the expression of [H] have been proposed by many researchers in order to take into account the previous aspects, see [23]. In particular, the second form of matrix [H] shown in Eq. (18) takes into account on one side the parabolic variation of the shear stresses through the thickness of each lamina (which satisfies the

local equilibrium) and on the other side the fact that such stresses have to vanish on both top and bottom faces of the plate. However, this modified form of [H] does not take into account the continuity of the shear stresses at the interfaces of the plies. For a deeper insight on such aspects the reader is addressed to [23].

It can be noticed that, when passing from the lamina material frame Γ to the laminate global frame Γ^{I}, the terms of the matrix $[\mathrm{Q}(\delta_k)]$ behave like those of a fourth-rank elasticity-like tensor, while the components of $[\widehat{\mathrm{Q}}(\delta_k)]$ behave like those of a second-rank symmetric tensor, see [4, 19]. Therefore $[\mathrm{Q}(\delta_k)]$ and $[\widehat{\mathrm{Q}}(\delta_k)]$ can be expressed (within the laminate global frame) by means of the polar formalism as follows:

$$\begin{array}{lclclclclcl}
\mathrm{Q}_{xx} & = & & T_0 & + & 2T_1 & + & R_0\cos 4(\Phi_0+\delta_k) & + & 4R_1\cos 2(\Phi_1+\delta_k) & , \\
\mathrm{Q}_{xy} & = & - & T_0 & + & 2T_1 & - & R_0\cos 4(\Phi_0+\delta_k) & & & , \\
\mathrm{Q}_{xs} & = & & & & & & R_0\sin 4(\Phi_0+\delta_k) & + & 2R_1\sin 2(\Phi_1+\delta_k) & , \\
\mathrm{Q}_{yy} & = & & T_0 & + & 2T_1 & + & R_0\cos 4(\Phi_0+\delta_k) & - & 4R_1\cos 2(\Phi_1+\delta_k) & , \\
\mathrm{Q}_{ys} & = & & & & & - & R_0\sin 4(\Phi_0+\delta_k) & + & 2R_1\sin 2(\Phi_1+\delta_k) & , \\
\mathrm{Q}_{ss} & = & & T_0 & & & - & R_0\cos 4(\Phi_0+\delta_k) & & & ,
\end{array} \tag{19}$$

and

$$\begin{array}{lclcl}
\widehat{\mathrm{Q}}_{qq} & = & T & + & R\cos 2(\Phi+\delta_k)\ , \\
\widehat{\mathrm{Q}}_{qr} & = & & & R\sin 2(\Phi+\delta_k)\ , \\
\widehat{\mathrm{Q}}_{rr} & = & T & - & R\cos 2(\Phi+\delta_k)\ .
\end{array} \tag{20}$$

To be remarked that in the previous equations it is the material frame of the k-th lamina (and not the global one) which is turned counter-clock wise by an angle δ_k around the x_3 axis. In Eqs. (19) and (20) T_0, T_1, R_0, R_1, Φ_0 and Φ_1 are the polar parameters of the in-plane reduced stiffness tensor of the lamina, while T, R, and Φ are those of the reduced out-of-plane stiffness tensor: all of these parameters solely depend upon the ply material properties (e.g. if the ply is orthotropic the polar parameters of $[\mathrm{Q}(\delta_k)]$ depend upon E_1, E_2, G_{12} and ν_{12}, while those of $[\widehat{\mathrm{Q}}(\delta_k)]$ depend upon G_{23} and G_{13}).

In order to better analyse and understand the mechanical response of the laminate it is useful to homogenise the units of the matrices [A], [B], [D] and [H] to those of the ply reduced stiffness matrices as follows:

$$\begin{array}{ll}
[\mathrm{A}^*] = \dfrac{1}{h}[\mathrm{A}] & , \\
[\mathrm{B}^*] = \dfrac{2}{h^2}[\mathrm{B}] & , \\
[\mathrm{D}^*] = \dfrac{12}{h^3}[\mathrm{D}] & , \\
[\mathrm{H}^*] = \begin{cases} \dfrac{1}{h}[\mathrm{H}] \quad \text{(basic)} & , \\ \dfrac{12}{5h}[\mathrm{H}] \quad \text{(modified)} & . \end{cases}
\end{array} \tag{21}$$

In the framework of the polar formalism it is possible to express also matrices [A*], [B*], [D*] and [H*] in terms of their polar parameters. In particular the homogenised membrane, membrane/bending coupling and bending stiffness matrices behave like a fourth-rank elasticity-like tensor while the homogenised shear matrix behaves like a second-rank symmetric tensor. Moreover, the polar parameters of these matrices can be expressed as functions of the polar parameters of the lamina reduced stiffness matrices and of the geometrical properties of the stack (i.e. layer orientation and position). The polar representation of [A*], [B*] and [D*] is (see [19]):

$$
\begin{aligned}
T_{0A^*} &= T_0 \quad , \\
T_{1A^*} &= T_1 \quad , \\
R_{0A^*}e^{\mathrm{i}4\Phi_{0A^*}} &= \frac{1}{n}R_0 e^{\mathrm{i}4\Phi_0}\sum_{k=1}^{n} e^{\mathrm{i}4\delta_k} \quad , \\
R_{1A^*}e^{\mathrm{i}2\Phi_{1A^*}} &= \frac{1}{n}R_1 e^{\mathrm{i}2\Phi_1}\sum_{k=1}^{n} e^{\mathrm{i}2\delta_k} \quad ,
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
T_{0B^*} &= 0 \quad , \\
T_{1B^*} &= 0 \quad , \\
R_{0B^*}e^{\mathrm{i}4\Phi_{0B^*}} &= \frac{1}{n^2}R_0 e^{\mathrm{i}4\Phi_0}\sum_{k=1}^{n} b_k e^{\mathrm{i}4\delta_k} \quad , \\
R_{1B^*}e^{\mathrm{i}2\Phi_{1B^*}} &= \frac{1}{n^2}R_1 e^{\mathrm{i}2\Phi_1}\sum_{k=1}^{n} b_k e^{\mathrm{i}2\delta_k} \quad ,
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
T_{0D^*} &= T_0 \quad , \\
T_{1D^*} &= T_1 \quad , \\
R_{0D^*}e^{\mathrm{i}4\Phi_{0D^*}} &= \frac{1}{n^3}R_0 e^{\mathrm{i}4\Phi_0}\sum_{k=1}^{n} d_k e^{\mathrm{i}4\delta_k} \quad , \\
R_{1D^*}e^{\mathrm{i}2\Phi_{1D^*}} &= \frac{1}{n^3}R_1 e^{\mathrm{i}2\Phi_1}\sum_{k=1}^{n} d_k e^{\mathrm{i}2\delta_k} \quad ,
\end{aligned}
\tag{24}
$$

while that of matrix [H*] (see Appendix A) can be stated as:

$$
\begin{aligned}
T_{H^*} &= \begin{cases} T & \text{(basic)} \quad , \\ 2T & \text{(modified)} \quad , \end{cases} \\
R_{H^*}e^{\mathrm{i}2\Phi_{H^*}} &= \begin{cases} \dfrac{1}{n}Re^{\mathrm{i}2\Phi}\displaystyle\sum_{k=1}^{n} e^{\mathrm{i}2\delta_k} & \text{(basic)} \quad , \\ \dfrac{1}{n^3}Re^{\mathrm{i}2\Phi}\displaystyle\sum_{k=1}^{n} (3n^2 - d_k)e^{\mathrm{i}2\delta_k} & \text{(modified)} \quad , \end{cases}
\end{aligned}
\tag{25}
$$

From Eqs. (22)-(25) it seems that, at the macro-scale, the laminate behaviour is governed by a set of 21 polar parameters: six for each one of the matrices [A*], [B*] and [D*],

whilst three for the shear stiffness matrix. In this set the isotropic moduli of [B*] are null, whilst those of [A*], [D*] and [H*] are identical (or proportional) to the isotropic moduli of the layer reduced stiffness matrices. The only polar parameters which depend upon the geometrical properties of the stack (i.e. orientation angles and positions of the plies) are the anisotropic moduli and polar angles of [A*], [B*] and [D*] together with the deviatoric modulus and polar angle of [H*] for an overall number of 14 polar parameters which can be designed (by acting on the geometric parameters of the stacking sequence) in order to achieve the desired mechanical response for the laminate at the macro-scale. However, as it is detailed in Appendix B, the deviatoric modulus and the polar angle of matrix [H*] can be expressed (depending on the considered formulation for [H*]) as a linear combination of the anisotropic polar modulus R_1 and the related polar angle Φ_1 of matrices [A*] and [D*] as follows:

$$R_{H^*}e^{\mathrm{i}2\Phi_{H^*}} = \begin{cases} R_{1A^*}\dfrac{R}{R_1}e^{\mathrm{i}2(\Phi_{1A^*}+\Phi-\Phi_1)} & \text{(basic)} \quad , \\ \dfrac{R}{R_1}e^{\mathrm{i}2(\Phi-\Phi_1)}\left(3R_{1A^*}e^{\mathrm{i}2\Phi_{1A^*}} - R_{1D^*}e^{\mathrm{i}2\Phi_{1D^*}}\right) & \text{(modified)} \quad . \end{cases} \tag{26}$$

Eq. (26) means that (when the material of the elementary ply is fixed *a priori*) the overall mechanical response of the laminate depends only on the anisotropic polar moduli and the related polar angles of matrices [A*], [B*] and [D*] even in the framework of the First-order Shear Deformation Theory. In particular the number of polar parameters to be designed remains unchanged when passing from the context of CLT to that of FSDT: the designer can act (through a variation of geometric parameters such as layers orientations and positions) only on the anisotropic polar moduli and polar angles of the membrane, membrane/bending coupling and bending stiffness matrices, the deviatoric modulus and the polar angle of the shear stiffness matrix being directly linked to them. Moreover, as it clearly appears from the first expression of Eq.(26), when using the basic definition of the laminate shear stiffness matrix, the ratio between the deviatoric part of the matrix [H*], i.e. $R_{H^*}e^{\mathrm{i}2\Phi_{H^*}}$, and the anisotropic term $R_{1A^*}e^{\mathrm{i}2\Phi_{1A^*}}$ of matrix [A*] is constant once the material of the constitutive layer is chosen: such a ratio does not depend upon the layers orientations and positions, rather it solely varies with the material properties of the constitutive layer (i.e. when varying the polar parameters R_1, Φ_1, R, Φ).

As a conclusive remark of this section, it is noteworthy that since in almost all of the real-life engineering applications the designers look for an uncoupled laminate (i.e. [B*] = [O]), the total number of laminate parameters reduces from 12 to eight. In addition, by means of the polar formalism it is possible to further reduce the total number of laminate parameters to be conceived: when using *quasi-homogeneous* laminates [19, 20],

i.e. laminates which satisfy the following properties ([C*] is the homogeneity matrix)

$$\begin{aligned} [\mathrm{B}^*] &= [\mathrm{O}] \quad , \\ [\mathrm{C}^*] &= [\mathrm{A}^*] - [\mathrm{D}^*] = [\mathrm{O}] \quad , \end{aligned} \tag{27}$$

the total number of laminate polar parameters reduces from eight to four. The only quantities to be conceived are the anisotropic polar moduli and the related polar angles of the laminate membrane stiffness matrix (or the bending one since they are identical), namely R_{0A^*}, R_{1A^*}, Φ_{0A^*}, Φ_{1A^*} and this result generally applies even when stating the laminate design problem in the framework of the FSDT (and not only within that of the CLT).

4 Elastic symmetries of the laminate: the Polar Approach in the framework of the FSDT

In this Section the problem of designing the elastic symmetries of a laminate will be briefly recalled. As described by Vannucci in [18], such a problem can be stated as an unconstrained minimisation problem in the space of the laminate polar parameters. However, the classical formulation presented in [18] (later modified and extended to the case of laminates with variable number of plies in [8, 19]), which currently relies on the use of the CLT hypotheses, will be here extended to the theoretical framework of the FSDT.

Before introducing the unified formulation for the design problem of the elastic symmetries of a laminate it is opportune to make some comments about all the possible elastic symmetries of the stiffness matrices describing the behaviour of the laminate in the context of the FSDT. In particular, as in the case of the CLT, the membrane, membrane/bending coupling and bending stiffness matrices can show one among the four different elastic symmetries of a fourth-rank elasticity-like tensor, as described in Section 2 (i.e. ordinary orthotropy, R_0-orthotropy, square symmetry and isotropy).

Concerning the laminate out-of-plane shear stiffness matrix (since its components behave like those of a second-rank symmetric tensor) it can be characterised only by a unique symmetry: the isotropy (when the deviatoric polar modulus of this matrix is null). In any other case this matrix is always orthotropic. However, as stated in the previous Section, the polar parameters of such a matrix, depending on the considered formulation, can always be obtained as a linear combination of the polar parameters of matrices [A*] and [D*]. As a consequence, the elastic symmetries of matrix [H*] closely depend upon those of [A*] and [D*]. After a quick glance to Eq. (26) and according to the considered formulation for the laminate shear stiffness matrix (basic or modified) the following remarks about the elastic symmetries of [H*] can be deduced.

1. In the case of the basic formulation, matrix [H*] is isotropic *if and only if* the laminate membrane stiffness matrix [A*] shows a square symmetric behaviour, i.e.:

$$R_{H^*} = 0 \Leftrightarrow R_{1A^*} = 0 \ . \tag{28}$$

2. In the case of the modified formulation, a *sufficient condition* for obtaining the isotropy of the laminate out-of-plane shear stiffness matrix is that both matrices [A*] and [D*] must be characterised by a square elastic symmetry. Conversely, if [H*] is isotropic the laminate membrane and bending stiffness matrices are not necessarily characterised by a square-symmetric behaviour:

$$\begin{aligned} & R_{1A^*} = R_{1D^*} = 0 \Rightarrow R_{H^*} = 0 \qquad , \\ & \text{but} \ \ R_{H^*} = 0 \not\Rightarrow R_{1A^*} = R_{1D^*} = 0 \quad . \end{aligned} \tag{29}$$

3. If the laminate has the same elastic response in membrane and bending, i.e. [A*] = [D*], when using the enriched formulation for [H*], the previous condition becomes also a necessary condition. In other words the following two-way relationship applies:

$$\text{if} \ \ [\mathrm{C}^*] = [\mathrm{O}] \ \ \text{then} \ \ R_{H^*} = 0 \Leftrightarrow R_{1A^*} = R_{1D^*} = 0 \ . \tag{30}$$

Let us introduce now the problem of designing the laminate elastic behaviour. Such a problem consists in finding at least one-stacking sequence meeting the desired set of elastic symmetries for the laminate (e.g. membrane/bending uncoupling, membrane orthotropy, bending isotropy, etc.). When using the polar formalism and when considering the theoretical framework of the FSDT such a problem can be stated as an unconstrained minimisation problem as follows:

$$\min_{\delta_1,...,\delta_n} \Psi\left(\delta_1, ..., \delta_n\right) = \{\mathrm{f}\}^{\mathrm{T}} \left[\mathrm{W}\right] \{\mathrm{f}\} \ , \tag{31}$$

where Ψ is the overall objective function expressing the desired laminate behaviour and δ_k is the k-th layer orientation ($k = 1, ...n$). {f} is the vector of the partial objective functions (each one linked to a particular elastic symmetry of the laminate) while $[W]$ is a positive semi-definite diagonal matrix of weights whose terms can be equal to either zero or one (depending on the considered combination of elastic symmetries). The components of the vector {f} as well as the related physical meaning are listed here below:

- $\mathrm{f}_1 = \dfrac{\| [\mathrm{B}^*] \|}{\| [\mathrm{Q}] \|}$ represents the membrane/bending uncoupling condition;
- $\mathrm{f}_2 = \dfrac{\| [\mathrm{C}^*] \|}{\| [\mathrm{Q}] \|}$ represents the homogeneity condition;

- $f_3 = \dfrac{\Phi_{0A^*} - \Phi_{1A^*} - K_{A^*}}{\pi/4}$ with $K_{A^*} = 0, 1$ represents the ordinary orthotropy condition for [A*];
- $f_4 = \dfrac{R_{0A^*}}{R_0}$ representing the R_0-orthotropy condition for [A*];
- $f_5 = \dfrac{R_{1A^*}}{R_1}$ representing the square symmetry condition for [A*];
- $f_6 = \dfrac{\sqrt{R_{0A^*}^2 + 4R_{1A^*}^2}}{\sqrt{R_0^2 + 4R_1^2}}$ representing the isotropy condition for [A*];
- $f_7 = \dfrac{\Phi_{0D^*} - \Phi_{1D^*} - K_{D^*}}{\pi/4}$ with $K_{D^*} = 0, 1$ represents the ordinary orthotropy condition for [D*];
- $f_8 = \dfrac{R_{0D^*}}{R_0}$ representing the R_0-orthotropy condition for [D*];
- $f_9 = \dfrac{R_{1D^*}}{R_1}$ representing the square symmetry condition for [D*];
- $f_{10} = \dfrac{\sqrt{R_{0D^*}^2 + 4R_{1D^*}^2}}{\sqrt{R_0^2 + 4R_1^2}}$ representing the isotropy condition for [D*];
- $f_{11} = \dfrac{\Phi_{0D^*} - \Phi_{0A^*}}{\pi/4}$ represents the coincidence of the main orhtotropy axes in the case of the square symmetry for both membrane and bending stiffness matrices;
- $f_{12} = \dfrac{\Phi_{1D^*} - \Phi_{1A^*}}{\pi/4}$ represents the coincidence of the main orhtotropy axes in the case of the ordinary orhtotropy or R_0-orthotropy for both membrane and bending stiffness matrices;
- $f_{13} = \dfrac{R_{H^*}}{R}$ representing the isotropy condition for [H*];
- $f_{14} = \dfrac{R_{0A^*} - \widehat{R}_{0A^*}}{\widehat{R}_{0A^*}}$ represents a condition on the value of the first anisotropic modulus for [A*] which can be used in the cases of ordinary orthotropy or square symmetry (but not in the cases of both R_0-orthotropy and isotropy);
- $f_{15} = \dfrac{R_{1A^*} - \widehat{R}_{1A^*}}{\widehat{R}_{1A^*}}$ representing a condition on the value of the second anisotropic modulus for [A*] which can be used in the cases of ordinary orthotropy or R_0-orthotropy (but not in the cases of both square symmetry and isotropy);
- $f_{16} = \dfrac{\Phi_{1A^*} - \widehat{\Phi}_{1A^*}}{\pi/4}$ representing a condition on the value of the orientation of the main orthotropy axis for [A*] which can be used in the cases of ordinary orthotropy or R_0-orthotropy (but not in the cases of both square symmetry and isotropy);

- $\mathrm{f}_{17} = \dfrac{\Phi_{0A^*} - \widehat{\Phi}_{0A^*}}{\pi/4}$ representing a condition on the value of the orientation of the main orthotropy axis for [A*] which can be used in the case of square symmetry (but not in the cases of ordinary orthotropy, R_0-orthotropy and isotropy);

- $\mathrm{f}_{18} = \dfrac{R_{0D^*} - \widehat{R}_{0D^*}}{\widehat{R}_{0D^*}}$ represents a condition on the value of the first anisotropic modulus for [D*] which can be used in the cases of ordinary orthotropy or square symmetry (but not in the cases of both R_0-orthotropy and isotropy);

- $\mathrm{f}_{19} = \dfrac{R_{1D^*} - \widehat{R}_{1D^*}}{\widehat{R}_{1D^*}}$ represents a condition on the value of the second anisotropic modulus for [D*] which can be used in the cases of ordinary orthotropy or R_0-orthotropy (but not in the cases of both square symmetry and isotropy);

- $\mathrm{f}_{20} = \dfrac{\Phi_{1D^*} - \widehat{\Phi}_{1D^*}}{\pi/4}$ representing a condition on the value of the orientation of the main orthotropy axis for [D*] which can be used in the cases of ordinary orthotropy or R_0-orthotropy (but not in the cases of both square symmetry and isotropy);

- $\mathrm{f}_{21} = \dfrac{\Phi_{0D^*} - \widehat{\Phi}_{0D^*}}{\pi/4}$ representing a condition on the value of the orientation of the main orthotropy axis for [D*] which can be used in the case of square symmetry (but not in the cases of ordinary orthotropy, R_0-orthotropy and isotropy).

It can be noticed that all of the components of the vector {f} are expressed in terms of the polar parameters of the laminate stiffness matrices and that they have been normalised with the corresponding counterparts of the ply stiffness matrices, i.e. [Q] and [$\widehat{\mathrm{Q}}$]. Moreover, the expression of the matrix norm used for the first two partial functions is that proposed by Kandil and Verchery [24]:

$$\| [\mathrm{Q}] \| = \sqrt{T_0^2 + 2T_1^2 + R_0^2 + 4R_1^2} \ , \tag{32}$$

an analogous relationship applies for matrices [B*] and [C*]. Of course, the terms belonging to the diagonal of the weight matrix [W] cannot be all different from zero at the same time: for instance it is not possible to have a laminate which is simultaneously orthotropic and isotropic in membrane, or a laminate which is quasi-homogeneous orthotropic in membrane and isotropic in bending (indeed if the laminate is quasi-homogeneous it is characterised by the same elastic behaviour in membrane and bending), etc. Therefore a particular care must be taken in tuning the terms of the weight matrix.

As a conclusive remark it is noteworthy that the objective function Ψ is a dimensionless, positive semi-definite convex function in the space of laminate polar parameters, since it is defined as a sum of convex functions, see Eq. (31). Nevertheless, such a function is highly non-convex in the space of plies orientation angles because the laminate polar parameters

depend upon circular functions of these angles, see Eqs. (22)-(25). Finally, one of the advantages of such a formulation consists in the fact that the absolute minima of Ψ are known a priori since they are the zeroes of this function. For more details about the nature of this problem the reader is addressed to [8, 19].

5 Studied cases and results

In this Section some meaningful numerical examples concerning the problem of designing the laminate elastic behaviour will be illustrated in order to numerically check the validity of the analytical results for the elastic symmetries of the laminate out-of-plane shear stiffness matrix presented in Eqs. (28)-(30). Moreover, such examples will also show on one hand the effectiveness of using the polar approach in the framework of the FSDT, while on the other hand it will be (numerically) proved the existence of some non-conventional stacking sequences satisfying a given set of elastic requirements imposed on the homogenised stiffness matrices of the laminate, i.e. [A*], [B*], [D*] and [H*]. In particular, in the following subsections the problem of designing the laminate elastic symmetries is formulated and solved in the following cases:

- an uncoupled laminate with square symmetric membrane and isotropic out-of-plane shear behaviours (basic formulation);
- an uncoupled laminate with an isotropic out-of-plane shear behaviour (modified formulation);
- a quasi-homogeneous laminate with square symmetric membrane-bending and isotropic out-of-plane shear behaviours (modified formulation).

Since the elastic behaviour of the laminate depends upon the elastic properties of the constitutive lamina, the results must refer to a given material. In the case of the numerical examples illustrated in this Section a transverse isotropic unidirectional carbon/epoxy ply has been chosen, whose material properties are listed in Table 1. In addition the number of layers n composing the laminated plate was fixed equal to 16.

Due to the nature of the optimisation problem of Eq. (31), i.e. a highly non-convex unconstrained minimisation problem in the space of the layers orientations, the new version of the genetic algorithm BIANCA [12, 19, 25] has been employed to find a solution. In this case, each individual has a genotype composed of n chromosomes, i.e. one for each ply, characterised by a single gene coding the layer orientation. It must be pointed out that the orientation angle of each lamina can get all the values in the range $[-89°, 90°]$ with a discretisation step of 1°. Such a discretisation step has been chosen in order to prove that laminates with given elastic properties can be easily obtained by abandoning the well-known conventional rules for tailoring the laminate stack (e.g. symmetric-balanced

stacks) which extremely shrink the search space for the problem at hand. Therefore, the true advantages in using non-conventional staking sequences are at least two: on one hand when using such a discretisation step for the plies orientations it is possible to explore the overall design space of problem (31), while on the other hand the polar-genetic approach leads to find very general stacks (nor symmetric neither balanced) that fully meet the elastic properties with a fewer number of plies (hence lighter) than the standard ones. For more details about these aspects the reader is addressed to [8, 19].

Finally, regarding the value of the genetic parameters for the GA BIANCA, used to solve the unconstrained minimisation problem (31), they are listed in Table 2. For more details on the numerical techniques developed within the new version of BIANCA and the meaning of the values of the different parameters tuning the GA the reader is addressed to [19, 25].

5.1 Case 1: uncoupled laminate with square symmetric membrane and isotropic out-of-plane shear behaviours (basic formulation)

Concerning the mathematical formulation of the constitutive law, the basic formulation has been employed in this example for expressing the out-of-plane shear stiffness matrix of the laminate. The aim of this first case is to design an uncoupled laminate showing a square symmetric membrane stiffness matrix. Therefore, by imposing this kind of symmetry on matrix $[\mathrm{A}^*]$ the designer can automatically obtain an isotropic out-of-plane shear stiffness matrix, as a consequence of Eq. (28). Equivalently, when using the basic formulation for matrix $[\mathrm{H}^*]$, by imposing the isotropy condition on this matrix the elastic requirement on the square symmetry of the laminate membrane stiffness matrix is fully met. In this case, the expression of the overall objective function Ψ of Eq. (31) is composed only by the sum of two quadratic functions and it can be obtained in two different but equivalent ways:

- as the sum of the square of functions f_1 and f_5 by setting $\mathrm{W}_{11} = \mathrm{W}_{55} = 1$ and $\mathrm{W}_{ii} = 0,\ (i = 2, ..., 21 \text{ with } i \neq 5)$, i.e.

$$\Psi = {\mathrm{f}_1}^2 + {\mathrm{f}_5}^2 = \left(\frac{\| [\mathrm{B}^*] \|}{\| [\mathrm{Q}] \|}\right)^2 + \left(\frac{R_{1A^*}}{R_1}\right)^2 \ ; \tag{33}$$

- as the sum of the square of functions f_1 and f_{13} by setting $\mathrm{W}_{11} = \mathrm{W}_{1313} = 1$ and $\mathrm{W}_{ii} = 0,\ (i = 2, ..., 21 \text{ with } i \neq 13)$, i.e.

$$\Psi = {\mathrm{f}_1}^2 + {\mathrm{f}_{13}}^2 = \left(\frac{\| [\mathrm{B}^*] \|}{\| [\mathrm{Q}] \|}\right)^2 + \left(\frac{R_{H^*}}{R}\right)^2 \ . \tag{34}$$

Table 3 shows two examples of laminate stacking sequences satisfying the criteria of Eqs. (33)-(34). The residual in the last column is the value of the objective function Ψ for each solution (recall that exact solutions correspond to zeros of the objective function). As in each numerical technique the "true" solution always lies within a small numerical interval

of tolerance in the neighbourhood of the exact one: this tolerance is exactly the residual. A discussion on the importance of the numerical residual in this type of problems can be found in [18]. It can be noticed that the optimal stacking sequences are really general: they are nor symmetric neither balanced and they fully meet the elastic symmetry requirements imposed on the laminate through Eq. (33) or (34) with only 16 plies.

Table 4 lists the value of the laminate polar parameters for the best stacking sequence (solution n. 1) of Table 3, while Fig. 1 illustrates the related polar diagrams of both the first component for matrices [A*], [B*] and [D*] and those of [H*] (when using the basic formulation). One can notice that, according to the theoretical result of Eq. (28), the laminate is characterised both by a square symmetric membrane stiffness behaviour (whose main orthotropic axis is oriented at $-18°$, see Table 3) and by an isotropic out-of-plane shear elastic response. In addition the laminate is practically uncoupled ($B^*{}_{xx}$ reduces to a small point in the centre of the plot) while it is completely anisotropic in bending because no elastic requirements have been imposed on [D*]. It is noteworthy that such results have been found with very general stacks composed of a few number of plies: it is really difficult (if not impossible) to obtain the same laminate mechanical response with standard multilayer plates, i.e. plates characterised by a symmetric, balanced lay-up.

As a final remark, Fig. 2 shows the variation of the value of the objective function of the best solution (of Table 3) along generations for problem (31) for this first case. One can easily see that the optimum solution has been found only after 160 generations. Since the problem is highly non-convex, at the end of the genetic calculation it is possible to find within the population not only the best solution but also some fitting quasi-optimal solution like the solution n.2 illustrated in Table 3: the presence of such solutions (whereof solution n.2 is only an example among others composing the final population) can be effectively exploited by the designer which wants to deeply investigate their mechanical response with respect to different design criteria (e.g. buckling, natural frequencies, etc.).

5.2 Case 2: uncoupled laminate with an isotropic out-of-plane shear behaviour (modified shear matrix)

For this second case, concerning the laminate constitutive law, the enriched formulation has been considered to express the matrix [H*]. Here, the goal is to design an uncoupled laminate with an isotropic out-of-plane shear elastic response. Therefore, due to the theoretical result of Eq. (29), the laminate will not necessarily be characterised by any special elastic symmetry for both membrane and bending behaviours.

In this case, the expression of the overall objective function Ψ of Eq. (31) is composed only of the sum of two quadratic functions and it can be easily obtained by setting $W_{11} =$

$\mathrm{W}_{1313} = 1$ and $\mathrm{W}_{ii} = 0,\ (i = 2, ..., 21$ with $i \neq 13)$:

$$\Psi = \mathrm{f_1}^2 + \mathrm{f_{13}}^2 = \left(\frac{\| [\mathrm{B}^*] \|}{\| [\mathrm{Q}] \|}\right)^2 + \left(\frac{R_{H^*}}{R}\right)^2 . \tag{35}$$

Two examples of laminate stacking sequences satisfying the criteria of Eq. (35) are listed in Table 3. Table 5 lists the value of the laminate polar parameters for the best stacking sequence (solution n. 1) of Table 3, while Fig. 3 illustrates the related polar diagrams for matrices [A*], [B*], [D*] and [H*] (when using the modified formulation). One can notice that, according to the theoretical result of Eq. (29), the laminate is characterised only by an isotropic out-of-plane shear elastic response. In this case the laminate is uncoupled ($\mathrm{B}^*{}_{xx}$ reduces to a small point in the centre of the plot) while it is completely anisotropic in both membrane and bending because, when using the modified form of matrix [H*], an isotropic out-of-plane shear behaviour does not necessarily imply a square symmetric behaviours for matrices [A*] and [D*].

Finally, Fig. 4 shows the variation of the value of the objective function for the best solution (of Table 3) along generations for problem (31) for this second case. It can be noticed that the optimum solution has been found after 185 generations. For the rest, the considerations already done for case 1 can be repeated here.

5.3 Case 3: quasi-homogeneous laminate with square symmetric membrane-bending and isotropic out-of-plane shear behaviours (modified shear matrix)

Even in this last case the modified formulation has been employed to express the out-of-plane shear stiffness matrix of the laminate. The aim of this example is the design of a quasi-homogeneous laminate with a fully square symmetric elastic behaviour (both in extension and bending) and with the main axis of symmetry (for [A*] and [D*]) oriented at $\widehat{\Phi}_{0A^*} = \widehat{\Phi}_{0D^*} = 0°$. Moreover, due to the theoretical result of Eq. (30), when the laminate is homogeneous and characterised by a square symmetric elastic response it will also show an isotropic out-of-plane shear behaviour.

In this case, the expression of the overall objective function Ψ of Eq. (31) can be obtained by setting $\mathrm{W}_{11} = \mathrm{W}_{22} = \mathrm{W}_{55} = \mathrm{W}_{1717} = 1$ and $\mathrm{W}_{ii} = 0,\ (i = 3, ..., 21$ with $i \neq 5, 17)$:

$$\Psi = \mathrm{f_1}^2 + \mathrm{f_2}^2 + \mathrm{f_5}^2 + \mathrm{f_{17}}^2 = \left(\frac{\| [\mathrm{B}^*] \|}{\| [\mathrm{Q}] \|}\right)^2 + \left(\frac{\| [\mathrm{C}^*] \|}{\| [\mathrm{Q}] \|}\right)^2 + \left(\frac{R_{1A^*}}{R_1}\right)^2 + \left(\frac{\Phi_{0A^*} - \widehat{\Phi}_{0A^*}}{\pi/4}\right)^2 . \tag{36}$$

Two examples of laminate stacking sequences satisfying the criteria of Eq. (36) are listed in Table 3: also in this case the optimal stacks are very general stacks. Table 6 lists the value of the laminate polar parameters for the best stacking sequence (solution n. 1) of Table 3, while Fig. 5 illustrates the related polar diagrams for matrices [A*], [B*], [D*] and

[H*]. One can notice that, according to the theoretical result of Eq. (30), the laminate is characterised both by a full square symmetric elastic response (matrices [A*] and [D*]) and by an isotropic out-of-plane shear behaviour. Moreover the laminate is quasi-homogeneous, i.e. uncoupled and with the same homogenised membrane and bending behaviour. Finally, the main axis of symmetry for both matrices [A*] and [D*] is oriented at 0°.

As a final remark of this section, Fig. 6 shows the variation of the value of the objective function for the best solution (of Table 3) along generations for problem (31) for this last case: the optimum solution has been found after about 125 generations. For the rest, the considerations already done for cases 1 and 2 can be repeated here.

6 Conclusions

In this work the Verchery's polar method for representing plane tensors has been extended and employed within the conceptual framework of the First-order Shear Deformation Theory of laminates. The following major results were analytically derived.

1. The number of independent tensor invariants characterising the mechanical response of the laminate remains unchanged when passing from the context of the CLT to that of the FSDT.

2. The elastic symmetries of the laminate out-of-plane shear stiffness matrix depend upon those of membrane and bending stiffness matrices: in particular, depending on the considered formulation, the isotropic behaviour of the laminate shear stiffness matrix is closely related to the square symmetric behaviour of the membrane stiffness matrix (basic formulation) or to the square symmetry of both bending and membrane elastic response (modified formulation).

3. The unified formulation of the problem of designing the laminate elastic symmetries has been modified and extended to the context of the FSDT.

To the best of the author's knowledge, this is the first time that a mathematical formulation based upon tensor invariants (namely the polar method) has been applied to the conceptual framework of the FSDT. The mechanical response of the laminated plate is represented by means of the polar formalism that offers several advantages: a) the polar invariants are directly linked to the tensor elastic symmetries, b) the polar method allows for eliminating from the procedure redundant mechanical properties and c) it lets to easily express the change of reference frame.

The effectiveness of the proposed approach has been proved both analytically and numerically by means of some novel and meaningful numerical examples. The numerical results presented in this works show that when the well-known hypotheses and rules for

tailoring laminates are abandoned (i.e. by using symmetric, balanced stacks and by considering a small set of layer orientations shrunk to the values 0°, 45° and 90°) it is possible to design laminates with enhanced elastic and (more generally) mechanical responses, very difficult (if not impossible) to be obtained otherwise.

Finally, it is opinion of the author that the polar-genetic approach can be extended also to the theoretical framework of more accurate theories such as the Third-order Shear Deformation Theory or even higher order theories coupled with equivalent single layer kinematic models: research is ongoing on this topics.

Appendix A Determination of the polar parameters of matrix $[\mathrm{H}^*]$

Since the components of matrix $[\mathrm{H}^*]$ behave like those of a second-rank symmetric tensor, its polar representation (expressed within the laminate global frame Γ^{I}), according to Eq. (4), writes:

$$\begin{aligned} T_{H^*} &= \frac{\mathrm{H}^*{}_{qq} + \mathrm{H}^*{}_{rr}}{2} \quad , \\ R_{H^*} e^{\mathrm{i}2\Phi_{H^*}} &= \frac{\mathrm{H}^*{}_{qq} - \mathrm{H}^*{}_{rr}}{2} + \mathrm{i}\mathrm{H}^*{}_{qr} \quad . \end{aligned} \tag{A. 1}$$

Depending on the considered formulation for expressing matrix $[\mathrm{H}^*]$, its Cartesian components can be written in terms of those of the lamina out-of-plane stiffness matrix $[\widehat{\mathrm{Q}}]$ as:

$$\mathrm{H}^*{}_{ij} = \begin{cases} \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} \widehat{\mathrm{Q}}_{ij}(\delta_k) & \text{(basic)} \quad , \\ \dfrac{1}{n^3}\displaystyle\sum_{k=1}^{n} (3n^2 - d_k)\widehat{\mathrm{Q}}_{ij}(\delta_k) & \text{(modified)} \quad , \end{cases} \qquad (i,j = q,r) \ . \tag{A. 2}$$

Let us consider the expression of the isotropic modulus T_{H^*} of Eq. (A. 1). By injecting the expression of $\mathrm{H}^*{}_{qq}$ and $\mathrm{H}^*{}_{rr}$ given by Eq. (A. 2) we have:

$$T_{H^*} = \begin{cases} \dfrac{1}{2n}\displaystyle\sum_{k=1}^{n} \left[\widehat{\mathrm{Q}}_{qq}(\delta_k) + \widehat{\mathrm{Q}}_{rr}(\delta_k)\right] & \text{(basic)} \quad , \\ \dfrac{1}{2n^3}\displaystyle\sum_{k=1}^{n} (3n^2 - d_k)\left[\widehat{\mathrm{Q}}_{qq}(\delta_k) + \widehat{\mathrm{Q}}_{rr}(\delta_k)\right] & \text{(modified)} \quad . \end{cases} \tag{A. 3}$$

In order to obtain the expression of the isotropic modulus T_{H^*} in terms of the polar parameters of the out-of-plane shear stiffness matrix of the lamina, it suffices to inject the expression of $\widehat{\mathrm{Q}}_{qq}(\delta_k)$ and $\widehat{\mathrm{Q}}_{rr}(\delta_k)$ given by Eq. (20). After some standard algebraic passages and by considering the following equality

$$\sum_{k=1}^{n} (3n^2 - d_k) = 2n^3 \ , \tag{A. 4}$$

one can write the following expression:

$$T_{H^*} = \begin{cases} \dfrac{1}{2n}\sum_{k=1}^{n} 2T = T & \text{(basic)} \quad , \\ \dfrac{1}{2n^3}\sum_{k=1}^{n}(3n^2 - d_k)2T = 2T & \text{(modified)} \quad . \end{cases} \tag{A. 5}$$

Let us now consider the expression of the deviatoric part $R_{H^*}e^{\mathrm{i}2\Phi_{H^*}}$ of the laminate shear stiffness matrix given by Eq. (A. 1). By injecting the expression of $\mathrm{H}^*{}_{qq}$, $\mathrm{H}^*{}_{rr}$ and $\mathrm{H}^*{}_{qr}$ given by Eq. (A. 2) we have:

$$R_{H^*}e^{\mathrm{i}2\Phi_{H^*}} = \begin{cases} \dfrac{1}{n}\sum_{k=1}^{n}\left[\dfrac{\widehat{\mathrm{Q}}_{qq}(\delta_k) - \widehat{\mathrm{Q}}_{rr}(\delta_k)}{2} + \mathrm{i}\widehat{\mathrm{Q}}_{qr}(\delta_k)\right] & \text{(basic)} \quad , \\ \dfrac{1}{n^3}\sum_{k=1}^{n}(3n^2 - d_k)\left[\dfrac{\widehat{\mathrm{Q}}_{qq}(\delta_k) - \widehat{\mathrm{Q}}_{rr}(\delta_k)}{2} + \mathrm{i}\widehat{\mathrm{Q}}_{qr}(\delta_k)\right] & \text{(modified)} \quad , \end{cases} \tag{A. 6}$$

Consider now the polar expression of $\widehat{\mathrm{Q}}_{qq}(\delta_k)$, $\widehat{\mathrm{Q}}_{rr}(\delta_k)$ and $\widehat{\mathrm{Q}}_{qr}(\delta_k)$ given by Eq. (20). By injecting these relations in Eq. (A. 6) one obtains:

$$R_{H^*}e^{\mathrm{i}2\Phi_{H^*}} = \begin{cases} \dfrac{1}{n}\sum_{k=1}^{n}\left[R\cos 2\left(\Phi + \delta_k\right) + \mathrm{i}R\sin 2\left(\Phi + \delta_k\right)\right] & \text{(basic)} \quad , \\ \dfrac{1}{n^3}\sum_{k=1}^{n}(3n^2 - d_k)\left[R\cos 2\left(\Phi + \delta_k\right) + \mathrm{i}R\sin 2\left(\Phi + \delta_k\right)\right] & \text{(modified)} \quad , \end{cases} \tag{A. 7}$$

In order to derive the final form of the deviatoric part of matrix $[\mathrm{H}^*]$ it suffices to apply the following equality to Eq (A. 7):

$$\cos(\alpha + \beta) + \mathrm{i}\sin(\alpha + \beta) = e^{\mathrm{i}(\alpha+\beta)} = e^{\mathrm{i}\alpha}e^{\mathrm{i}\beta} \ . \tag{A. 8}$$

When applying the previous equality to Eq. (A. 7) we obtain:

$$R_{H^*}e^{\mathrm{i}2\Phi_{H^*}} = \begin{cases} \dfrac{1}{n}Re^{\mathrm{i}2\Phi}\sum_{k=1}^{n}e^{\mathrm{i}2\delta_k} & \text{(basic)} \quad , \\ \dfrac{1}{n^3}Re^{\mathrm{i}2\Phi}\sum_{k=1}^{n}(3n^2 - d_k)e^{\mathrm{i}2\delta_k} & \text{(modified)} \quad . \end{cases} \tag{A. 9}$$

Appendix B The link between the polar parameters of $[\mathrm{H}^*]$ and those of $[\mathrm{A}^*]$ and $[\mathrm{D}^*]$

In order to analytically derive the link between the deviatoric part of matrix $[\mathrm{H}^*]$ and the second anisotropic polar modulus R_1 and the related polar angle Φ_1 of matrices $[\mathrm{A}^*]$

and [D*], let us consider the expression of the quantities $\sum_{k=1}^{n} e^{\mathrm{i}2\delta_k}$ and $\sum_{k=1}^{n}(3n^2-d_k)e^{\mathrm{i}2\delta_k}$ appearing in Eq. (A. 9). These quantities actually depend upon the polar parameters of the membrane and bending stiffness matrices of the laminate. A quick glance to Eqs. (22) and (24) suffices to determine their expression. Indeed, from Eq. (22) we have:

$$\sum_{k=1}^{n} e^{\mathrm{i}2\delta_k} = \frac{nR_{1A^*}e^{\mathrm{i}2\Phi_{1A^*}}}{R_1 e^{\mathrm{i}2\Phi_1}} = n\frac{R_{1A^*}}{R_1}e^{\mathrm{i}2(\Phi_{1A^*}-\Phi_1)} \ , \tag{B. 1}$$

while from Eq. (24) we obtain:

$$\sum_{k=1}^{n} d_k e^{\mathrm{i}2\delta_k} = \frac{n^3 R_{1D^*}e^{\mathrm{i}2\Phi_{1D^*}}}{R_1 e^{\mathrm{i}2\Phi_1}} = n^3\frac{R_{1D^*}}{R_1}e^{\mathrm{i}2(\Phi_{1D^*}-\Phi_1)} \ . \tag{B. 2}$$

The expression of quantity $\sum_{k=1}^{n}(3n^2-d_k)e^{\mathrm{i}2\delta_k}$ can be obtained by combining Eqs. (B. 1) and (B. 2) as follows:

$$\sum_{k=1}^{n}(3n^2-d_k)e^{\mathrm{i}2\delta_k} = 3n^2\sum_{k=1}^{n}e^{\mathrm{i}2\delta_k} - \sum_{k=1}^{n}d_k e^{\mathrm{i}2\delta_k} = 3n^3\frac{R_{1A^*}}{R_1}e^{\mathrm{i}2(\Phi_{1A^*}-\Phi_1)} - n^3\frac{R_{1D^*}}{R_1}e^{\mathrm{i}2(\Phi_{1D^*}-\Phi_1)} \ . \tag{B. 3}$$

Finally, by substituting Eqs. (B. 1) and (B. 3) into Eq. (A. 9) (and after some standard passages) we can obtained the desired result:

$$R_{H^*}e^{\mathrm{i}2\Phi_{H^*}} = \begin{cases} \dfrac{1}{n}Re^{\mathrm{i}2\Phi}n\dfrac{R_{1A^*}}{R_1}e^{\mathrm{i}2(\Phi_{1A^*}-\Phi_1)} = R_{1A^*}\dfrac{R}{R_1}e^{\mathrm{i}2(\Phi_{1A^*}+\Phi-\Phi_1)} \ , \\ \dfrac{1}{n^3}Re^{\mathrm{i}2\Phi}\left[3n^3\dfrac{R_{1A^*}}{R_1}e^{\mathrm{i}2(\Phi_{1A^*}-\Phi_1)} - n^3\dfrac{R_{1D^*}}{R_1}e^{\mathrm{i}2(\Phi_{1D^*}-\Phi_1)}\right] = \\ = \dfrac{R}{R_1}e^{\mathrm{i}2(\Phi-\Phi_1)}\left(3R_{1A^*}e^{\mathrm{i}2\Phi_{1A^*}} - R_{1D^*}e^{\mathrm{i}2\Phi_{1D^*}}\right) \ . \end{cases} \tag{B. 4}$$

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Tables

Technical constants		Polar parameters of [Q]		Polar parameters of $[\widehat{\mathrm{Q}}]$	
E_1	161000 MPa	T_0	23793.3868 MPa	T	5095.4545 MPa
E_2	9000 MPa	T_1	21917.8249 MPa	R	1004.5454 MPa
G_{12}	6100 MPa	R_0	17693.3868 MPa	Φ	90 deg
ν_{12}	0.26	R_1	19072.0711 MPa		
ν_{23}	0.1	Φ_0	0 deg		
		Φ_1	0 deg		
Density and thickness					
ρ	1.58×10^{-6} Kg/mm^3				
t_{ply}	0.125 mm				

Table 1: Material properties of the carbon-epoxy lamina.

Genetic parameters	
N_{pop}	1
N_{ind}	500
N_{gen}	500
p_{cross}	0.85
p_{mut}	$1/N_{\mathrm{ind}}$
Selection	roulette-wheel
Elitism	active

Table 2: Genetic parameters of the GA BIANCA for problem (31).

Case N.	Solution N.	Stacking sequence	n	Residual
1	1	[64/-36/63/-72/4/-5/-5/81/-36/62/-13/85/40/-53/-13/70]	16	4.5742×10^{-7}
	2	[15/-89/-24/-63/8/62/60/-81/-13/-60/5/18/85/73/-52/6]	16	2.5810×10^{-6}
2	1	[-9/55/62/-21/47/-37/86/-57/52/-53/-2/37/-28/60/-14/64]	16	2.5693×10^{-6}
	2	[7/-33/-76/-25/83/64/-35/84/33/18/37/-71/-27/-10/-28/89]	16	6.8820×10^{-5}
3	1	[73/6/-58/26/-19/88/-29/89/-62/7/41/76/7/70/-6/-60]	16	5.0327×10^{-6}
	2	[87/-51/-1/55/23/-2/-12/-74/61/78/-66/7/-69/-30/70/12]	16	2.3628×10^{-5}

Table 3: Numerical results of problem (31) for cases 1, 2 and 3.

In plane elastic behaviour			
Polar parameters	[A*]	[B*]	[D*]
T_0 [MPa]	23793.3868	0	23793.3868
T_1 [MPa]	21917.8249	0	21917.8249
R_0 [MPa]	7089.4990	28.2753	8714.2147
R_1 [MPa]	0.3627	13.2899	3313.7496
Φ_0 [deg]	-18	N.D.	-25
Φ_1 [deg]	N.D.	N.D.	77
Out-of-plane elastic behaviour			
Polar parameters	[H*] (basic form)		
T [MPa]	5095.4545		
R [MPa]	0.0191		
Φ [deg]	N.D.		

Table 4: Laminate polar parameters for the best stacking sequence of case 1 (N.D.=not defined, i.e. meaningless for the considered combination of laminate elastic symmetries).

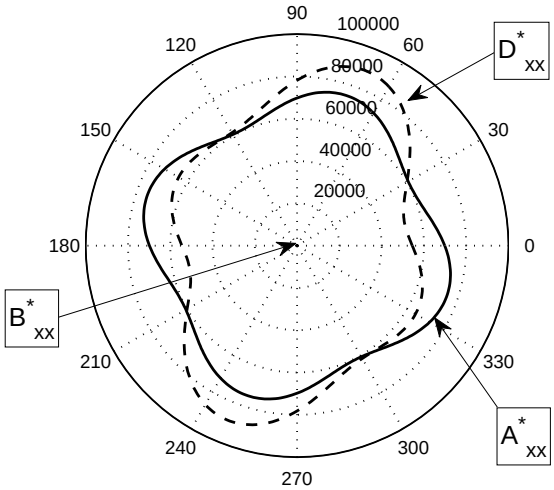
In plane elastic behaviour			
Polar parameters	[A*]	[B*]	[D*]
T_0 [MPa]	23793.3868	0	23793.3868
T_1 [MPa]	21917.8249	0	21917.8249
R_0 [MPa]	8389.0299	69.9780	12660.3816
R_1 [MPa]	1760.0603	19.9785	5266.3785
Φ_0 [deg]	-30	N.D.	-24
Φ_1 [deg]	31	N.D.	31
Out-of-plane elastic behaviour			
Polar parameters	[H*] (modified form)		
T [MPa]	10190.909		
R [MPa]	0.7772		
Φ [deg]	N.D.		

Table 5: Laminate polar parameters for the best stacking sequence of case 2 (N.D.=not defined, i.e. meaningless for the considered combination of laminate elastic symmetries).

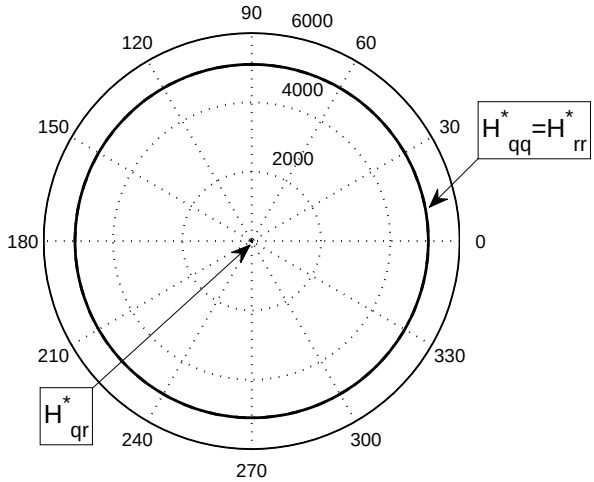
In plane elastic behaviour			
Polar parameters	[A*]	[B*]	[D*]
T_0 [MPa]	23793.3868	0	23793.3868
T_1 [MPa]	21917.8249	0	21917.8249
R_0 [MPa]	4200.7794	61.0565	4211.5750
R_1 [MPa]	23.3058	22.6314	49.0406
Φ_0 [deg]	0	N.D.	0
Φ_1 [deg]	N.D.	N.D.	N.D.
Out-of-plane elastic behaviour			
Polar parameters	[H*] (modified form)		
T [MPa]	10190.909		
R [MPa]	1.2434		
Φ [deg]	N.D.		

Table 6: Laminate polar parameters for the best stacking sequence of case 3 (N.D.=not defined, i.e. meaningless for the considered combination of laminate elastic symmetries).

Figures



a)



b)

Figure 1: a)First component of the laminate membrane, membrane/bending coupling and bending stiffness matrices and b) the three components of the laminate out-of-plane shear stiffness matrix, best solution of case 1.

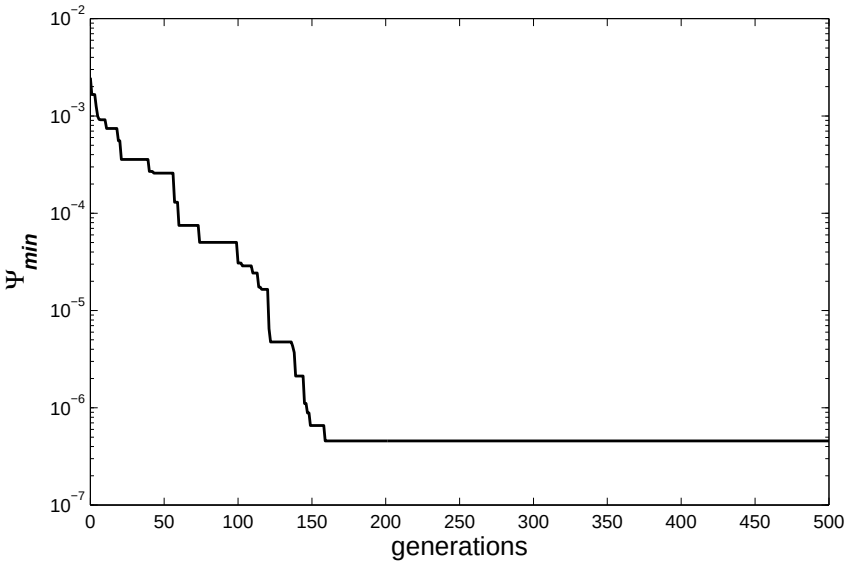
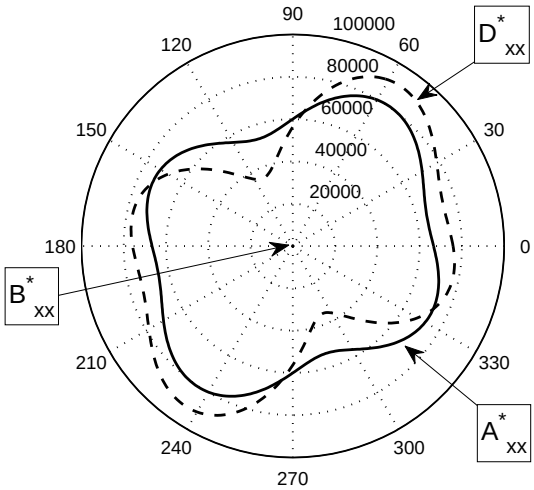


Figure 2: Best values of the objective function along generations, case 1.



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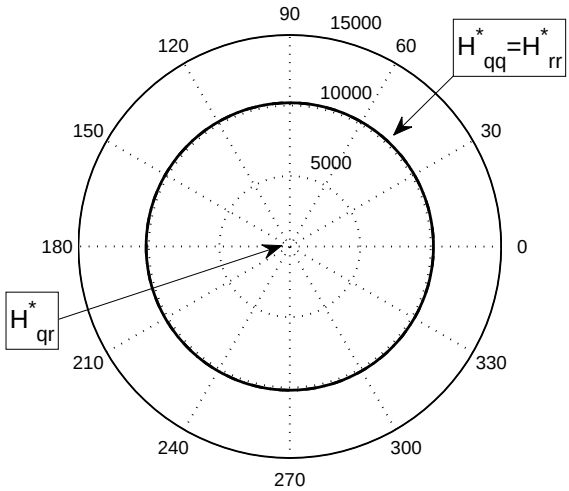


b)

Figure 3: a)First component of the laminate membrane, membrane/bending coupling and bending stiffness matrices and b) the three components of the laminate out-of-plane shear stiffness matrix, best solution of case 2.

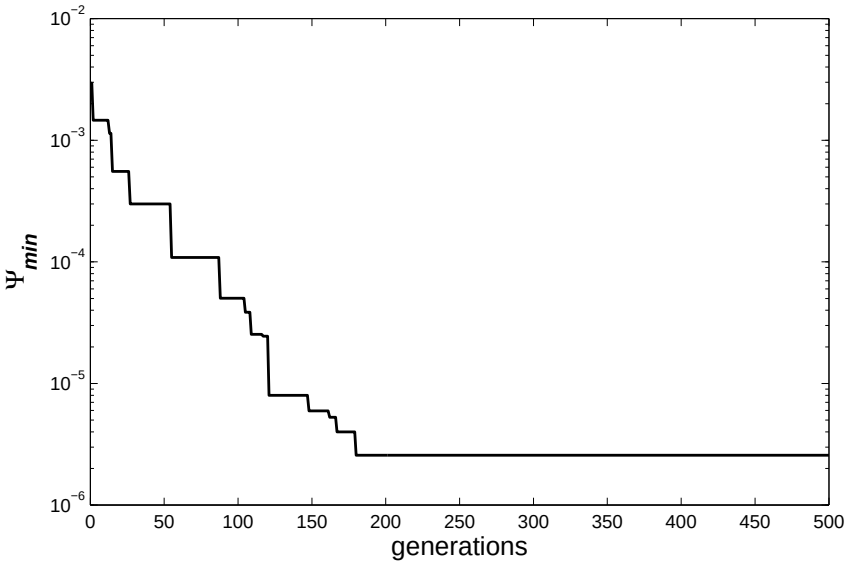
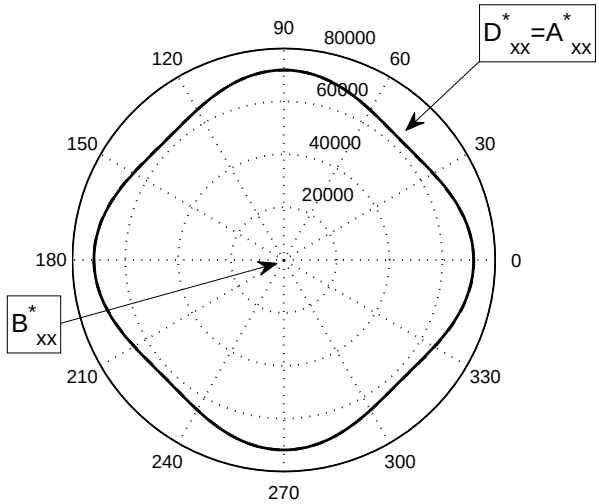
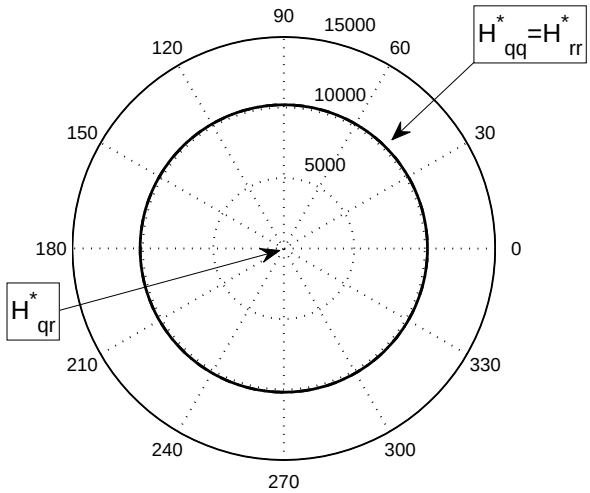


Figure 4: Best values of the objective function along generations, case 2.



a)



b)

Figure 5: a)First component of the laminate membrane, membrane/bending coupling and bending stiffness matrices and b) the three components of the laminate out-of-plane shear stiffness matrix, best solution of case 3.

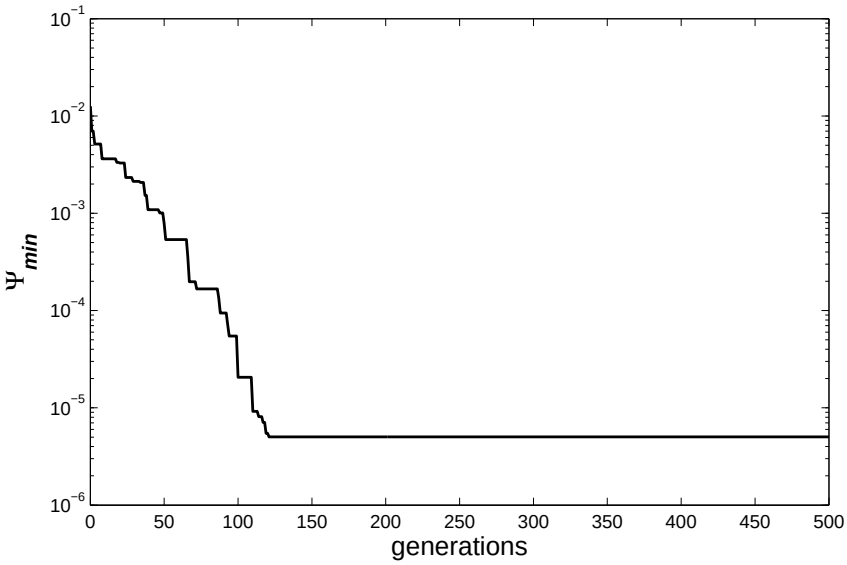


Figure 6: Best values of the objective function along generations, case 3.