On the correlation between stiffness and strength properties of anisotropic laminates

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Abstract

This paper focuses on the analytical formulation of a tensorial laminate-level failure criterion. The criterion is formulated and expressed in the framework of the first-order shear deformation theory (FSDT) in order to take into account the influence of the transverse shear stresses on the failure mechanisms. More precisely the most common polynomial ply-level failure criteria (expressed under a unified matrix formulation) are considered and reformulated at the laminate level. The proposed unified formulation relies on the utilisation of the polar formalism generalised to the FSDT framework. Through this approach all the considered criteria can be formulated in terms of tensor invariants. Furthermore, thanks to the polar representation, an important theoretical result is proven: the existence of a set of analytical relationships between the laminate strength and stiffness invariants.

Keywords:

Composites; Failure criteria; Anisotropy; Polar method; Strength; FSDT.

1 Introduction

The analysis and design of composite structures is a quite cumbersome multi-scale problem. The complexity is actually due to two intrinsic properties of composite materials, i.e. the heterogeneity and the anisotropy. Although the heterogeneity gets involved at all scales (micro, meso and macro), several homogenisation theories let simplify the description of the macroscopic mechanical response of composite structures. The simplest one is the Classical Laminate Theory (CLT). Nevertheless, the results obtained by using the CLT are not sufficiently accurate for those applications involving moderately thick (or thick) composite parts. To overcome this difficulty more accurate theories have been developed, e.g. the First-order Shear Deformation Theory (FSDT) [1].

The second intrinsic property of composite materials, i.e. the anisotropy, intervenes mainly at both meso-scale (that of the constitutive lamina) and macro-scale (that of the laminate). It is well known that the behaviour of an anisotropic continuum depend upon the direction. As a consequence a considerable number of independent mechanical parameters are needed to characterise the mechanical response of such a continuum.

Normally the Cartesian representation of tensors is employed to describe the behaviour of an anisotropic material, see [2]. While on one hand the Cartesian representation seems to be the “most natural” representation to describe the anisotropy, on the other hand it shows a major drawback: the material parameters depend upon the coordinate system chosen for characterising the mechanical response of the continuum. As a consequence, the anisotropy of the material is described by a set of parameters which are not (tensor) invariant quantities and that represent the response of the material only in a particular frame and not in a general one.

Several alternative analytical representations can be found in literature. Some of them rely on the use of tensor invariants which allow for describing the behaviour of an anisotropic continuum through intrinsic material quantities. Of course, such representations do not imply a reduction in the number of parameters needed to fully characterise the material behaviour. Nevertheless, since these intrinsic material quantities are tensor invariants on the one hand they allow to describe the mechanical response of the material regardless to the considered reference frame and on the other hand they let to better highlight some physical aspects that cannot be easily caught when using the Cartesian representation.

In the framework of the design of composite materials several analytical representations of (plane) anisotropy were developed in the past and among them the most commonly employed is that introduced by Tsai and Pagano [3]. The main drawbacks of this representation are basically three: firstly not all parameters are tensor invariants, secondly they do not have a simple and immediate physical meaning and, finally, they are not all independent, see [4].

In 1979 Verchery [5] introduced the polar method for representing fourth-rank elasticity-like plane tensors. This representation has been enriched and deeply studied later by Vannucci and his co-workers [6–10]. The polar method relies upon a complex variable transformation by taking inspiration from a classical technique often employed in analytical mechanics, see for instance the works of Kolosov [11] and Green and Zerna [12]. The main advantages of the polar formalism are essentially three: a) it is a representation of anisotropy which is based on tensor invariants, b) such invariants have an immediate physical meaning which is linked to the different (elastic) symmetries of the tensor and

c) the change of reference frame can be expressed in a straightforward way.

Concerning the problem of the design of a composite structure, the polar method has been applied in the framework of the CLT for different real-life engineering applications (see [13–19]) and recently extended to the FSDT [4, 20] and to the Third-order Shear Deformation Theory (TSDT) [21].

Anisotropy strongly affects both stiffness and strength of structures. The strength (or alternatively the weakness) of a given material is usually described by a *failure criterion*. The aim of a failure criterion is the evaluation of the *limit load* that the structure can withstand before the failure arises. We can separate the failure criteria into two distinct classes: the phenomenological ones [3, 22–24] and the physically-based ones [25]. When using phenomenological failure criteria, the occurrence of the failure is checked through the computation of a scalar indicator, i.e. the *failure index*: only a single condition must be verified. Nevertheless, no indication is given about the mechanism of failure that has been activated. Conversely, physically-based failure criteria check separately multiple failure mechanisms which are supposed as independent thus, the uniqueness of the failure index is lost.

Several failure criteria have been developed for composites materials, see [26]. A very exhaustive assessment of failure criteria for composite laminates has been done in the World Wide Failure Exercise (WWFE) proposed by Hinton, Soden and Kaddour in [27–29]. All the failure criteria discussed in [27–29] are ply-level failure criteria, i.e. they are checked for each layer composing the laminate in order to determine the so-called *first-play failure*. Nevertheless, unlike the description of the elasticity which is realised at each scale (micro, meso and macro) the description of the strength for anisotropic materials and structures is usually done at (and limited to) both microscopic and mesoscopic scales.

Indeed it is quite hard (and unusual too) to find research studies dealing with the problem of the homogenisation of the strength properties of a composite material at the macroscopic scale (i.e. at the laminate level). The problem of the formulation of an equivalent laminate-level failure criterion is addressed only in few works, [30–32]. In [30] De Buhan presented a study on the strength homogenisation of a generic composite material. He considered a heterogeneous continuum composed of two constituents. The strength homogenisation was evaluated at both mesoscopic and macroscopic scales. The proposed model is very simple and makes use of the mixture law: at the macroscopic scale the strength is the average of the strengths of each constitutive phase, weighted by the corresponding material volume fraction. In [31] De Buhan and Taliercio addressed the same problem by extending the theoretical model proposed in [30] and by considering among the constitutive phases the presence of the fibre-matrix interface. The strength domain is assumed to vary point-wise and it is approximated by an equivalent homogenised strength field composed of two parts: the isotropic part, that does not depends upon the volume fraction, and the anisotropic one depending on the volume fraction of each phase.

In [9] the most common phenomenological failure criteria (i.e. the Hill [22], Hoffman [23], Tsai-Wu [24] and the Zhang-Evans [33] criteria) have been formulated in the mathematical framework of the polar method. This unified formulation through invatiants has been utilised to formulate and solve the problem of maximising the strength of a generic orthotropic sheet in terms of its material orientation. In [19] the unified formulation presented in [9] was generalised in order to formulate a homogenised failure criterion at the laminate level. The Tsai-Hill criterion was formulated (at the laminate level) in terms of the laminate strength invariants and the problem of maximising the strength of a laminated plate subject to in-plane loads was addressed.

The present study represents a further generalisation of the unified approach proposed in [19]: here tensorial laminate-level failure criterion is reformulated and expressed in the framework of the FSDT in order to take into account also the out-of-plane shear stresses that can lead to the failure of the laminate. Moreover in this work the most common failure criteria of Tsai-Hill, Hoffman, Tsai-Wu and Zhang-Evans are considered and reformulated at the laminate level. Finally thanks to the analytical results of the polar analysis of the FSDT [4,20] an important theoretical result is proven: the existence of a set of analytical relationships between the laminate strength and stiffness invariants. The manuscript is organised as follows: Section 2 briefly recalls the fundamentals of the polar method while Section 3 describes the main results of the polar formalism applied to FSDT. In Section

4 the most common phenomenological failure criteria are formulated in the framework of the unified approach based on tensor invariants at the ply-level, while in Section 5 their laminate-level homogenised counterpart is discussed. Section 6 ends the paper with some concluding remarks and future perspectives.

2 Fundamentals of the Polar Method

In this section the main results of the Polar Method introduced by Verchery [5] are briefly recalled. The polar method is substantially a mathematical technique that allows for expressing any n-rank plane tensor through a set of tensor invariants. Inspired by the work of Green and Zerna [12], Verchery makes use of a (very classical) mathematical technique based upon a complex variable transformation in order to easily represent the affine transformation (in this case a rotation) of a plane tensor after a change of reference frame. For more details about the genesis of the polar method the reader is addressed to [6]. Here below only the main results concerning the polar representation of both second-rank symmetric plane tensors and fourth-rank elasticity-like (i.e. having both major and minor symmetries) plane tensors are briefly recalled.

In the framework of the polar formalism a second-rank symmetric plane tensor $Z_{ij},\ (i,j = 1,2)$, within the local frame $\varUpsilon = \{0; x_1, x_2, x_3\}$, can be stated as:

$$\begin{aligned} Z_{11} &= T + R\cos 2\varPhi \ , \\ Z_{12} &= \phantom{T + {}} R\sin 2\varPhi \ , \\ Z_{22} &= T - R\cos 2\varPhi \ , \end{aligned} \tag{1}$$

where T is the isotropic modulus, R the deviatoric one and $\varPhi$ the polar angle. From Eq. (1) it can be noticed that the three independent Cartesian components of a second-rank plane symmetric tensor are expressed in terms of three polar parameters: among them only two are tensor invariants, i.e. T and R, while the last one, namely the polar angle $\varPhi$, is needed to fix the reference frame. The converse relations are:

$$
\begin{aligned}
T &= \frac{Z_{11}+Z_{22}}{2} \quad , \\
Re^{\mathrm{i}2\varPhi} &= \frac{Z_{11}-Z_{22}}{2} + \mathrm{i}Z_{12} \quad ,
\end{aligned}
\tag{2}
$$

where $\mathrm{i}=\sqrt{-1}$ is the imaginary unit. For a second-rank plane tensor the only possible symmetry is the isotropy which can be obtained when the deviatoric modulus of the tensor is null, i.e. $R=0$. Furthermore, when using the polar formalism the components of the second-rank tensor can be expressed in a very straightforward manner in the frame $\varUpsilon^{\mathrm{I}}=\{0;x,y,z\}$ (turned counter-clock wise by an angle θ around the x_3 axis) as follows:

$$
\begin{aligned}
Z_{xx} &= T + R\cos 2(\varPhi-\theta) \ , \\
Z_{xy} &= R\sin 2(\varPhi-\theta) \ , \\
Z_{yy} &= T - R\cos 2(\varPhi-\theta) \ .
\end{aligned}
\tag{3}
$$

Indeed the change of frame can be easily obtained by simply subtracting the angle θ from the polar angle $\varPhi$.

Concerning a fourth-rank elasticity-like plane tensor L_{ijkl}, $(i,j,k,l=1,2)$ (expressed within the local frame $\varUpsilon$), its polar representation writes:

$$
\begin{aligned}
L_{1111} &= T_0 + 2T_1 + R_0\cos 4\varPhi_0 + 4R_1\cos 2\varPhi_1 \quad , \\
L_{1122} &= -T_0 + 2T_1 - R_0\cos 4\varPhi_0 \quad , \\
L_{1112} &= R_0\sin 4\varPhi_0 + 2R_1\sin 2\varPhi_1 \quad , \\
L_{2222} &= T_0 + 2T_1 + R_0\cos 4\varPhi_0 - 4R_1\cos 2\varPhi_1 \quad , \\
L_{2212} &= -R_0\sin 4\varPhi_0 + 2R_1\sin 2\varPhi_1 \quad , \\
L_{1212} &= T_0 - R_0\cos 4\varPhi_0 \quad .
\end{aligned}
\tag{4}
$$

As it clearly appears from Eq. (4) the six independent Cartesian components of L_{ijkl} are expressed in terms of six polar parameters: T_0 and T_1 are the isotropic moduli, R_0 and R_1 are the anisotropic ones, while $\varPhi_0$ and $\varPhi_1$ are the polar angles. Only five quantities are tensor invariants, namely the polar moduli T_0, T_1, R_0, R_1 together with the angular difference $\varPhi_0 - \varPhi_1$. One of the two polar angles, $\varPhi_0$ or $\varPhi_1$, can be arbitrarily chosen to fix the reference frame. The converse relationships are:

$$\begin{aligned}
8T_0 &= L_{1111} - 2L_{1122} + 4L_{1212} + L_{2222} \quad ,\\
8T_1 &= L_{1111} + 2L_{1122} + L_{2222} \quad ,\\
8R_0 e^{\mathrm{i}4\varPhi_0} &= L_{1111} - 2L_{1122} - 4L_{1212} + L_{2222} + 4\mathrm{i}(L_{1112} - L_{2212}) \quad ,\\
8R_1 e^{\mathrm{i}2\varPhi_1} &= L_{1111} - L_{2222} + 2\mathrm{i}(L_{1112} + L_{2212}) \quad .
\end{aligned} \tag{5}$$

Once again, thanks to the polar formalism it is very easy to express the Cartesian components of the fourth-rank tensor in the frame $\varUpsilon^{\mathrm{I}}$, in fact it suffices to subtract the angle θ from the polar angles $\varPhi_0$ and $\varPhi_1$ as follows:

$$\begin{aligned}
L_{xxxx} &= T_0 + 2T_1 + R_0\cos 4(\varPhi_0 - \theta) + 4R_1\cos 2(\varPhi_1 - \theta) \quad ,\\
L_{xxyy} &= -T_0 + 2T_1 - R_0\cos 4(\varPhi_0 - \theta) \quad ,\\
L_{xxxy} &= R_0\sin 4(\varPhi_0 - \theta) + 2R_1\sin 2(\varPhi_1 - \theta) \quad ,\\
L_{yyyy} &= T_0 + 2T_1 + R_0\cos 4(\varPhi_0 - \theta) - 4R_1\cos 2(\varPhi_1 - \theta) \quad ,\\
L_{yyxy} &= -R_0\sin 4(\varPhi_0 - \theta) + 2R_1\sin 2(\varPhi_1 - \theta) \quad ,\\
L_{xyxy} &= T_0 - R_0\cos 4(\varPhi_0 - \theta) \quad .
\end{aligned} \tag{6}$$

In the case of a fourth-rank elasticity-like tensor the polar invariants are directly linked to the (elastic) symmetries of the tensor, thus having an immediate physical meaning. Indeed the polar formalism offers an *algebraic* characterisation of the elastic symmetries. In particular it can be proved that for a fourth-rank elasticity-like plane tensor four different types of elastic symmetry exist:

- *Ordinary orthotropy*: this symmetry corresponds to the algebraic condition

$$\varPhi_0 - \varPhi_1 = K\frac{\pi}{4}, \quad K = 0, 1 \ . \tag{7}$$

  Indeed, for the same set of tensor invariants, i.e. T_0, T_1, R_0, R_1, two different shapes of orthotropy exist, depending on the value of K. Vannucci [6] shows that they correspond to the so-called *low* ($K = 0$) and *high* ($K = 1$) shear modulus orthotropic materials firstly studied by Pedersen [34]. However, this classification is rather limiting since the difference between these two classes of orhtotropy concerns, more generally, the global mechanical response of the material, see [6, 9].

- R_0-*Orthotropy*: the algebraic condition to attain this "special" orthotropy is

$$R_0 = 0 \ . \tag{8}$$

  In this case the Cartesian components of the fourth-rank tensor L_{ijkl} change (as a result of a frame rotation) as those of a second-rank tensor, see Eqs. (1),(4). The existence of this particular orthotropy has been found also for the 3D case [35].

- *Square symmetry*: it can be obtained by imposing the following condition

$$R_1 = 0 \ . \tag{9}$$

  This symmetry represents the 2D case of the well-known 3D cubic syngony.

- *Isotropy*: the fourth-rank elasticity-like tensor is isotropic when its anisotropic moduli are null, i.e. when the following condition is satisfied

$$R_0 = R_1 = 0 \ . \tag{10}$$

3 The Polar Formalism applied to the First-order Shear Deformation Theory of laminates

For sake of simplicity in this section all of the equations governing the laminate mechanical response will be formulated in the context of the Voigt's (matrix) notation. The passage from tensor notation to Voigt's notation can be easily expressed by the following two-way relationships among indexes (for both local and global frames):

$$\begin{aligned} \{11, 22, 33, 32, 31, 21\} \quad &\Leftrightarrow \quad \{1, 2, 3, 4, 5, 6\} \quad , \\ \{xx, yy, zz, zy, zx, yx\} \quad &\Leftrightarrow \quad \{x, y, z, q, r, s\} \quad . \end{aligned} \tag{11}$$

Let us consider a multilayer plate composed of n identical layers (i.e. layers having same material properties and thickness). Let be δ_k the orientation angle of the k-th ply ($k = 1, ..., n$), t_{ply} the thickness of the elementary lamina and $h = nt_{\text{ply}}$ the overall thickness of the plate. In the framework of the FSDT theory [1] the constitutive law of the laminated plate (expressed within the global frame of the laminate Υ^{I}) can be stated as:

$$\left\{ \begin{array}{c} \{\text{N}\} \\ \\ \{\text{M}\} \end{array} \right\} = \left[\begin{array}{cc} [\text{A}] & [\text{B}] \\ \\ [\text{B}] & [\text{D}] \end{array} \right] \left\{ \begin{array}{c} \{\varepsilon_0\} \\ \\ \{\chi_0\} \end{array} \right\} , \tag{12}$$

$$\{\text{F}\} = [\text{H}]\,\{\gamma_0\} \ , \tag{13}$$

where [A], [B] and [D] are the membrane, membrane/bending coupling and bending stiffness matrices of the laminate, while [H] is the out-of-plane shear stiffness matrix. {N}, {M} and {F} are the vectors of membrane forces, bending moments and shear forces per unit length, respectively, whilst $\{\varepsilon_0\}$, $\{\chi_0\}$ and $\{\gamma_0\}$ are the vectors of in-plane strains, curvatures and out-of-plane shear strains of the laminate middle plane, respectively. The expressions of matrices [A], [B] and [D] are:

$$\begin{aligned} [\text{A}] &= \frac{h}{n}\sum_{k=1}^{n} [\text{Q}\,(\delta_k)] \quad , \\ [\text{B}] &= \frac{1}{2}\left(\frac{h}{n}\right)^2 \sum_{k=1}^{n} b_k\,[\text{Q}\,(\delta_k)] \quad , \\ [\text{D}] &= \frac{1}{12}\left(\frac{h}{n}\right)^3 \sum_{k=1}^{n} d_k\,[\text{Q}\,(\delta_k)] \quad , \end{aligned} \tag{14}$$

with

$$\begin{aligned} b_k &= 2k - n - 1 \quad &, \quad \sum_{k=1}^{n} b_k = 0 \quad , \\ d_k &= 12k\,(k - n - 1) + 4 + 3n\,(n + 2) \quad &, \quad \sum_{k=1}^{n} d_k = n^3 \quad . \end{aligned} \tag{15}$$

It can be noticed that in Eq. (14) $[\mathrm{Q}(\delta_k)]$ is the in-plane reduced stiffness matrix of the k-th ply. Concerning Eq. (13), in literature one can find different expressions for the out-of-plane shear stiffness matrix of the laminate [H]. In the following it will be considered two different representations for this matrix, namely:

$$[\mathrm{H}] = \begin{cases} \dfrac{h}{n}\sum\limits_{k=1}^{n}[\widehat{\mathrm{Q}}(\delta_k)] & \text{(basic)} \quad , \\ \dfrac{5h}{12n^3}\sum\limits_{k=1}^{n}(3n^2 - d_k)[\widehat{\mathrm{Q}}(\delta_k)] & \text{(modified)} \quad . \end{cases} \tag{16}$$

In Eq. (16) $[\widehat{\mathrm{Q}}(\delta_k)]$ is the out-of-plane shear stiffness matrix of the elementary ply. The first form of the matrix [H] is the basic one wherein the shear stresses are constant through the thickness of each lamina. The second form of matrix [H] shown in Eq. (16) takes into account on the one side the parabolic variation of the shear stresses through the thickness of each lamina (which satisfies the local equilibrium) and on the other side the fact that such stresses have to vanish on both top and bottom faces of the plate. For a deeper insight on such aspects the reader is addressed to [1].

It can be proven that, when passing from the lamina material frame $\varUpsilon$ (which is turned counter-clock wise by the angle δ_k around the x_3 axis with respect to the laminate global frame) to the laminate global frame $\varUpsilon^{\mathrm{I}}$, the terms of the matrix $[\mathrm{Q}(\delta_k)]$ behave like those of a fourth-rank elasticity-like tensor, while the components of $[\widehat{\mathrm{Q}}(\delta_k)]$ behave like those of a second-rank symmetric tensor turned clockwise (although the rotation of the local frame is counter-clockwise), see [4, 20] for more details. Therefore $[\mathrm{Q}(\delta_k)]$ and $[\widehat{\mathrm{Q}}(\delta_k)]$ can be expressed (within the laminate global frame) by means of the polar formalism as follows:

$$\begin{aligned}
\mathrm{Q}_{xx} &= T_0 + 2T_1 + R_0\cos 4(\varPhi_0 + \delta_k) + 4R_1\cos 2(\varPhi_1 + \delta_k) \quad ,\\
\mathrm{Q}_{xy} &= -T_0 + 2T_1 - R_0\cos 4(\varPhi_0 + \delta_k) \quad ,\\
\mathrm{Q}_{xs} &= R_0\sin 4(\varPhi_0 + \delta_k) + 2R_1\sin 2(\varPhi_1 + \delta_k) \quad ,\\
\mathrm{Q}_{yy} &= T_0 + 2T_1 + R_0\cos 4(\varPhi_0 + \delta_k) - 4R_1\cos 2(\varPhi_1 + \delta_k) \quad ,\\
\mathrm{Q}_{ys} &= -R_0\sin 4(\varPhi_0 + \delta_k) + 2R_1\sin 2(\varPhi_1 + \delta_k) \quad ,\\
\mathrm{Q}_{ss} &= T_0 - R_0\cos 4(\varPhi_0 + \delta_k) \quad ,
\end{aligned} \tag{17}$$

and

$$
\begin{aligned}
\widehat{\mathrm{Q}}_{qq} &= T + R\cos 2(\varPhi - \delta_k) \quad , \\
\widehat{\mathrm{Q}}_{qr} &= R\sin 2(\varPhi - \delta_k) \quad , \\
\widehat{\mathrm{Q}}_{rr} &= T - R\cos 2(\varPhi - \delta_k) \quad .
\end{aligned}
\tag{18}
$$

In Eqs. (17) and (18) T_0, T_1, R_0, R_1, $\varPhi_0$ and $\varPhi_1$ are the polar parameters of the in-plane reduced stiffness tensor of the lamina, while T, R, and $\varPhi$ are those of the reduced out-of-plane stiffness tensor: all of these parameters solely depend upon the ply material properties (e.g. if the ply is orthotropic the polar parameters of $[\mathrm{Q}(\delta_k)]$ depend upon E_1, E_2, G_{12} and ν_{12}, while those of $[\widehat{\mathrm{Q}}(\delta_k)]$ depend upon G_{23} and G_{13}).

In order to better analyse and understand the mechanical response of the laminate it is useful to homogenise the units of the matrices [A], [B], [D] and [H] to those of the ply reduced stiffness matrices as follows:

$$
\begin{aligned}
[\mathrm{A}^*] &= \frac{1}{h}[\mathrm{A}] \quad , \\
[\mathrm{B}^*] &= \frac{2}{h^2}[\mathrm{B}] \quad , \\
[\mathrm{D}^*] &= \frac{12}{h^3}[\mathrm{D}] \quad , \\
[\mathrm{H}^*] &= \begin{cases} \dfrac{1}{h}[\mathrm{H}] & \text{(basic)} \quad , \\ \dfrac{12}{5h}[\mathrm{H}] & \text{(modified)} \quad . \end{cases}
\end{aligned}
\tag{19}
$$

In the framework of the polar formalism it is possible to express also matrices $[\mathrm{A}^*]$, $[\mathrm{B}^*]$, $[\mathrm{D}^*]$ and $[\mathrm{H}^*]$ in terms of their polar parameters. In particular the homogenised membrane, membrane/bending coupling and bending stiffness matrices behave like a fourth-rank elasticity-like tensor while the homogenised shear matrix behaves like a second-rank symmetric tensor. Moreover, the polar parameters of these matrices can be expressed as functions of the polar parameters of the lamina reduced stiffness matrices and of the geometrical properties of the stack (i.e. layer orientation and position). The polar representation of $[\mathrm{A}^*]$, $[\mathrm{B}^*]$ and $[\mathrm{D}^*]$ is (see [4,20]):

$$
\begin{aligned}
T_0^{A^*} &= T_0 &,\\
T_1^{A^*} &= T_1 &,\\
R_0^{A^*} e^{\mathrm{i}4\Phi_0^{A^*}} &= \frac{1}{n} R_0 e^{\mathrm{i}4\Phi_0} \sum_{k=1}^{n} e^{\mathrm{i}4\delta_k} &,\\
R_1^{A^*} e^{\mathrm{i}2\Phi_1^{A^*}} &= \frac{1}{n} R_1 e^{\mathrm{i}2\Phi_1} \sum_{k=1}^{n} e^{\mathrm{i}2\delta_k} &,
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
T_0^{B^*} &= 0 &,\\
T_1^{B^*} &= 0 &,\\
R_0^{B^*} e^{\mathrm{i}4\Phi_0^{B^*}} &= \frac{1}{n^2} R_0 e^{\mathrm{i}4\Phi_0} \sum_{k=1}^{n} b_k e^{\mathrm{i}4\delta_k} &,\\
R_1^{B^*} e^{\mathrm{i}2\Phi_1^{B^*}} &= \frac{1}{n^2} R_1 e^{\mathrm{i}2\Phi_1} \sum_{k=1}^{n} b_k e^{\mathrm{i}2\delta_k} &,
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
T_0^{D^*} &= T_0 &,\\
T_1^{D^*} &= T_1 &,\\
R_0^{D^*} e^{\mathrm{i}4\Phi_0^{D^*}} &= \frac{1}{n^3} R_0 e^{\mathrm{i}4\Phi_0} \sum_{k=1}^{n} d_k e^{\mathrm{i}4\delta_k} &,\\
R_1^{D^*} e^{\mathrm{i}2\Phi_1^{D^*}} &= \frac{1}{n^3} R_1 e^{\mathrm{i}2\Phi_1} \sum_{k=1}^{n} d_k e^{\mathrm{i}2\delta_k} &,
\end{aligned}
\tag{22}
$$

while that of matrix [H*] (see [4, 20]) can be stated as:

$$
\begin{aligned}
T^{H^*} &= \begin{cases} T & \text{(basic)} \quad ,\\ 2T & \text{(modified)} \quad , \end{cases}\\
R^{H^*} e^{\mathrm{i}2\Phi^{H^*}} &= \begin{cases} \dfrac{1}{n} R e^{\mathrm{i}2\Phi} \displaystyle\sum_{k=1}^{n} e^{-\mathrm{i}2\delta_k} & \text{(basic)} \quad ,\\ \dfrac{1}{n^3} R e^{\mathrm{i}2\Phi} \displaystyle\sum_{k=1}^{n} (3n^2 - d_k) e^{-\mathrm{i}2\delta_k} & \text{(modified)} \quad , \end{cases}
\end{aligned}
\tag{23}
$$

From Eqs. (20)-(23) it seems that, at the macroscopic scale, the laminate behaviour is governed by a set of 21 polar parameters: six for each one of the matrices [A*], [B*] and [D*], whilst three for the shear stiffness matrix. In this set the isotropic moduli of [B*] are null, whilst those of [A*], [D*] and [H*] are identical (or proportional) to the isotropic moduli of the layer reduced stiffness matrices. The only polar parameters which depend upon the geometrical properties of the stack (i.e. orientation angles and positions of the plies) are the anisotropic moduli and polar angles of [A*], [B*] and [D*] together with the deviatoric modulus and polar angle of [H*] for an overall number of 14 polar parameters which can be designed (by acting on the geometric parameters of the stacking sequence) in order to achieve the desired mechanical response for the laminate. However, as it is proven in [4, 20], the deviatoric modulus and the polar angle of matrix [H*] can be expressed (depending on the considered formulation for [H*]) as a linear combination of the anisotropic polar modulus R_1 and the related polar angle $\varPhi_1$ of matrices [A*] and [D*] as follows:

$$R^{H^*}e^{\mathrm{i}2\varPhi^{H^*}} = \begin{cases} R_1^{A^*}\dfrac{R}{R_1}e^{\mathrm{i}2(\varPhi+\varPhi_1-\varPhi_1^{A^*})} & \text{(basic)} \quad , \\ \dfrac{R}{R_1}e^{\mathrm{i}2(\varPhi+\varPhi_1)}\left(3R_1^{A^*}e^{-\mathrm{i}2\varPhi_1^{A^*}} - R_1^{D^*}e^{-\mathrm{i}2\varPhi_1^{D^*}}\right) & \text{(modified)} \quad , \end{cases} \tag{24}$$

Eq. (24) means that (when the material of the elementary ply is fixed *a priori*) the overall mechanical response of the laminate depends only on the anisotropic polar moduli and the related polar angles of matrices [A*], [B*] and [D*] even in the framework of the FSDT. In particular the number of polar parameters to be designed remains unchanged when passing from the context of CLT to that of FSDT. Moreover, as it clearly appears from the first expression of Eq.(24), when using the basic definition of the laminate shear stiffness matrix, the ratio between the deviatoric part of the matrix [H*], i.e. $R^{H^*}e^{\mathrm{i}2\varPhi^{H^*}}$, and the anisotropic term $R_1^{A^*}e^{\mathrm{i}2\varPhi_1^{A^*}}$ of matrix [A*] is constant once the material of the constitutive layer is chosen: such a ratio does not depend upon the layers orientations and positions, rather it solely varies with the material properties of the constitutive layer (i.e. when varying the polar parameters R_1, $\varPhi_1$, R, $\varPhi$). For a deeper insight in the matter the interested reader is addressed to [4, 20].

4 A unified formulation of ply-level failure criteria in the FSDT framework

In the following section we will present a generalised description of some phenomenological failure criteria, i.e. the Tsai-Hill, Hoffman, Tsai-Wu and Zhang-Evans criteria. All of these criteria are conceived mainly for orthotropic plies. As discussed in [9], it is possible to write all the stress-based criteria in a general matrix notation. We can express the Tsai-Hill, Hoffman and Tsai-Wu criteria by the general condition

$$F_{\dots} = \{\sigma\}^T [F] \{\sigma\} + \{\sigma\}^T \{f\} \leq 1 \ ; \tag{25}$$

where, in analogy with the tensorial formulation of Tsai-Wu, vectors $\{\sigma\}$ and $\{f\}$ behave like second-rank symmetric tensor, while matrix $[F]$ behave like a fourth-rank elasticity-like tensor. In the lamina material frame the components of $[F]$ and $\{f\}$ are:

$$[F] = \begin{bmatrix} F_{11} & F_{12} & F_{13} & 0 & 0 & 0 \\ F_{12} & F_{22} & F_{23} & 0 & 0 & 0 \\ F_{13} & F_{23} & F_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & F_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & F_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & F_{66} \end{bmatrix}, \{f\} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix} . \tag{26}$$

since the constitutive ply is assumed to be orthotropic, see [24]. The same quantities can be expressed in the global frame Υ^{I}:

$$[F] = \begin{bmatrix} F_{xx} & F_{xy} & F_{xz} & 0 & 0 & F_{xs} \\ F_{xy} & F_{yy} & F_{yz} & 0 & 0 & F_{ys} \\ F_{xz} & F_{yz} & F_{zz} & 0 & 0 & 0 \\ 0 & 0 & 0 & F_{qq} & F_{qr} & 0 \\ 0 & 0 & 0 & F_{qr} & F_{rr} & 0 \\ F_{xs} & F_{ys} & 0 & 0 & 0 & F_{ss} \end{bmatrix}, \{f\} = \begin{Bmatrix} f_x \\ f_y \\ f_z \\ 0 \\ 0 \\ f_s \end{Bmatrix} . \tag{27}$$

The expression of the components of $[F]$ and $\{f\}$ in terms of the lamina strength properties are given in [24]. As discussed in [9] matrix $[F]$ (in analogy with the compliance matrix $[S]$) can be interpreted as a *weakness matrix*.

According to the main hypotheses of the FSDT, the lamina is subject to a stress field with $\sigma_3 = \sigma_z = 0$. In this background the tensorial criterion of Eq. (25) writes:

$$F_{ply} = \left\{\sigma^{in}\right\}^T \left[F^{in}\right] \left\{\sigma^{in}\right\} + \left\{\sigma^{out}\right\}^T \left[F^{out}\right] \left\{\sigma^{out}\right\} + \left\{\sigma^{in}\right\}^T \left\{f^{in}\right\} \leq 1 \; ; \tag{28}$$

where:

$$\left\{\sigma^{in}\right\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{Bmatrix} \; , \left\{\sigma^{out}\right\} = \begin{Bmatrix} \sigma_q \\ \sigma_r \end{Bmatrix} \; , \tag{29}$$

and

$$\left[F^{in}\right] = \begin{bmatrix} F_{xx} & F_{xy} & F_{xs} \\ F_{xy} & F_{yy} & F_{ys} \\ F_{xs} & F_{ys} & F_{ss} \end{bmatrix} \; , \left[F^{out}\right] = \begin{bmatrix} F_{qq} & F_{qr} \\ F_{qr} & F_{rr} \end{bmatrix} \; , \left\{f^{in}\right\} = \begin{Bmatrix} f_x \\ f_y \\ f_s \end{Bmatrix} \; . \tag{30}$$

The generalised stress-based tensorial criterion can be expressed also in terms of strains, using the Hooke's law:

$$\begin{aligned} \left\{\sigma^{in}\right\} &= \left[Q^{in}\right] \left\{\varepsilon^{in}\right\} \; , \\ \left\{\sigma^{out}\right\} &= \left[Q^{out}\right] \left\{\varepsilon^{out}\right\} \; , \end{aligned} \tag{31}$$

with:

$$\left[Q^{in}\right] = \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{xy} & Q_{yy} & Q_{ys} \\ Q_{xs} & Q_{ys} & Q_{ss} \end{bmatrix} \; , \left[Q^{out}\right] = \begin{bmatrix} Q_{qq} & Q_{qr} \\ Q_{qr} & Q_{rr} \end{bmatrix} \; . \tag{32}$$

The criterion become:

$$F_{ply} = \left\{\varepsilon^{in}\right\}^T \left[G^{in}\right] \left\{\varepsilon^{in}\right\} + \left\{\varepsilon^{out}\right\}^T \left[G^{out}\right] \left\{\varepsilon^{out}\right\} + \left\{\varepsilon^{in}\right\}^T \left\{g^{in}\right\} \leq 1 \; ; \tag{33}$$

with:

$$
\begin{aligned}
&[G^{in}] = [Q^{in}]^T [F^{in}] [Q^{in}] \ , \\
&[G^{out}] = [Q^{out}]^T [F^{out}] [Q^{out}] \ , \\
&\{g^{in}\} = [Q^{in}]^T \{f^{in}\} \ .
\end{aligned}
\tag{34}
$$

The mathematical formulation of Eq. (33) correspond also to the strain-based failure criterion of Zhang-Evans [33] where $[G^{in}]$, $[G^{out}]$ and $\{g^{in}\}$ are the matrices and vector, respectively, of admissible strains of the material. In Eq. (33) where the failure criteria are expressed in terms of strains, matrices $[G^{in}]$ and $[G^{out}]$ can be considered as the analogous of the stiffness matrices $[Q^{in}]$ and $[Q^{out}]$, respectively; therefore matrices $[G^{in}]$ and $[G^{out}]$ can be considered as the *strength matrices* of the constitutive ply. In the framework of FSDT, the strain vectors are:

$$
\begin{aligned}
&\{\varepsilon^{in}\} = \{\varepsilon_0\} + z\,\{\chi_0\} \ , \\
&\{\varepsilon^{out}\} = \{\gamma_0\} \ .
\end{aligned}
\tag{35}
$$

By replacing Eqs. (35) in Eq. (33) the tensorial ply-level failure criterion in the framework of FSDT theory becomes:

$$
\begin{aligned}
F_{ply} = &\{\varepsilon_0\}^T [G^{in}] \{\varepsilon_0\} + z^2 \{\chi_0\}^T [G^{in}] \{\chi_0\} + 2z \{\varepsilon_0\}^T [G^{in}] \{\chi_0\} + \\
&\cdots + \{\gamma_0\}^T [G^{out}] \{\gamma_0\} + \{\varepsilon_0\}^T \{g^{in}\} + z\,\{\chi_0\}^T \{g^{in}\} \leq 1 \ .
\end{aligned}
\tag{36}
$$

5 Evaluation of the laminate strength using a homogenised criterion

In this Section a homogenised failure criterion that gives a measure of the strength of the laminate is formulated in the framework of the FSDT. Let us consider a multilayer plate with n plies. The generic k-th ply is characterised by the position of its bottom and top surfaces, z_{k-1} and z_k as shown in Fig. 1, the fibre orientation angle δ_k, the in-plane and out-of-plane reduced stiffness matrices $[Q^{in}(\delta_k)]$, $[Q^{out}(\delta_k)]$ and the strength matrices and vector $[G^{in}(\delta_k)]$, $[G^{out}(\delta_k)]$ and $\{g^{in}(\delta_k)\}$, respectively.

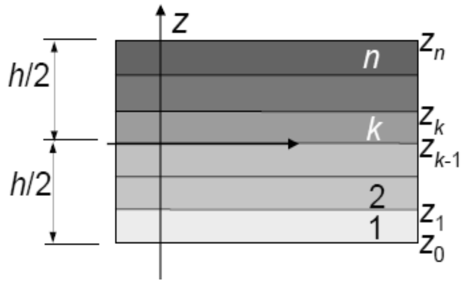

Figure 1: Sketch of the laminate layers and interfaces, see [36].

The "laminate failure index" is calculated by integrating Eq. (36) through the thickness of the plate:

$$F_{lam} = \frac{1}{h}\int_h F_{ply}(z)dz \leq 1. \tag{37}$$

Eq. (37) represents in some sense, an average strength that can be evaluated by composing, for a given strain field, the residual strength of each ply. Since the vectors of in-plane strain $\{\varepsilon_0\}$, curvature $\{\chi_0\}$ and transverse shear strain $\{\gamma_0\}$ do not depend upon the z coordinate, Eq. (37) becomes:

$$\begin{aligned} F_{lam} = \frac{1}{h}\Big[&\{\varepsilon_0\}^T \left(\textstyle\int_h [G^{in}]\, dz\right)\ \{\varepsilon_0\} + \{\chi_0\}^T \left(\textstyle\int_h [G^{in}]\, z^2 dz\right)\ \{\chi_0\} + \\ &2\{\varepsilon_0\}^T \left(\textstyle\int_h [G^{in}]\, z dz\right)\ \{\chi_0\} + \{\gamma_0\}^T \left(\textstyle\int_h [G^{out}]\, dz\right)\ \{\gamma_0\} + \\ &+\{\varepsilon_0\}^T \left(\textstyle\int_h \{g^{in}\}\, dz\right)\ + \{\chi_0\}^T \left(\textstyle\int_h \{g^{in}\}\, z dz\right)\Big] \leq 1\ . \end{aligned} \tag{38}$$

Integrating the previous matrices over the thickness of each constitutive layer and summing the different contributions leads to the following relationship:

$$\begin{aligned} F^{Lam}_{Hill} = \frac{1}{h}\Big(&\{\varepsilon_0\}^T \left[G^A\right]\{\varepsilon_0\} + \{\chi_0\}^T \left[G^D\right]\{\chi_0\} + 2\{\varepsilon_0\}^T \left[G^B\right]\{\chi_0\} + \\ &\{\gamma_0\}^T [G_H]\{\gamma 0\} + \{\varepsilon_0\}^T \{g_A\} + \{\chi_0\}^T \{g_D\}\Big) \leq 1. \end{aligned} \tag{39}$$

The different matrices of Eq. (39) are defined as:

$$
\begin{aligned}
[G_A] &= \sum_{k=1}^{n} \left[G^{in}\left(\delta_k\right)\right]\left(z_k - z_{k-1}\right), \\
[G_B] &= \frac{1}{2}\sum_{k=1}^{n} \left[G^{in}\left(\delta_k\right)\right]\left(z_k^2 - z_{k-1}^2\right), \\
[G_D] &= \frac{1}{3}\sum_{k=1}^{n} \left[G^{in}\left(\delta_k\right)\right]\left(z_k^3 - z_{k-1}^3\right), \\
[G_H] &= \sum_{k=1}^{n} \left[G^{out}\left(\delta_k\right)\right]\left(z_k - z_{k-1}\right), \\
\{g_A\} &= \sum_{k=1}^{n} \left\{g^{in}\left(\delta_k\right)\right\}\left(z_k - z_{k-1}\right), \\
\{g_D\} &= \frac{1}{2}\sum_{k=1}^{n} \left\{g^{in}\left(\delta_k\right)\right\}\left(z_k^2 - z_{k-1}^2\right).
\end{aligned}
\tag{40}
$$

where $[G_A]$, $\left[G^B\right]$, $\left[G^D\right]$ and $\left[G^H\right]$ are the laminate membrane, membrane/bending coupling, bending and shear strength matrices respectively, while $\{g_A\}$ and $\{g_D\}$ are the membrane and bending strength vectors related to the linear part of the failure criterion.

Eq. (39) represents the "Polynomial laminate-level failure criterion" for a multilayer plate modelled as an Equivalent Single Layer (ESL).

5.1 The polar analysis of the laminate strength properties

The polar analysis of the different matrices and vector appearing in Eq. (40) can be carried out through the same conceptual steps already presented and discussed in the case of the polar analysis of both first-order and third-order shear deformation theories [4,20,21]. In particular, when passing from the lamina material frame $\varUpsilon$ to the laminate global frame $\varUpsilon^{\mathrm{I}}$ (the frame $\varUpsilon$ is turned counter-clockwise by an angle δ_k around the x_3 axis):

- the terms of matrix $[G^{in}]$ behave like those of a fourth-rank elasticity-like tensor;
- the terms of matrix $[G^{out}]$ behave like those of a second-rank symmetric tensor turned clockwise (although the true rotation of the lamina reference system is counter-clockwise);
- the terms of vector $\{g^{in}\}$ behave like those of a second-rank symmetric tensor.

Therefore $[G^{in}]$, $[G^{out}]$ and $\{g^{in}\}$ can be expressed, in the laminate global frame $\varUpsilon^{\mathrm{I}}$, in terms of their polar parameters as:

$$
\begin{aligned}
\mathrm{G}^{in}_{xx} &= \varGamma_0^{in} + 2\varGamma_1^{in} + \varLambda_0^{in}\cos 4(\varOmega_0^{in}+\delta_k) + 4\varLambda_1^{in}\cos 2(\varOmega_1^{in}+\delta_k) \ ,\\
\mathrm{G}^{in}_{xy} &= -\varGamma_0^{in} + 2\varGamma_1^{in} - \varLambda_0^{in}\cos 4(\varOmega_0^{in}+\delta_k) \ ,\\
\mathrm{G}^{in}_{xs} &= \varLambda_0^{in}\sin 4(\varOmega_0^{in}+\delta_k) + 2\varLambda_1^{in}\sin 2(\varOmega_1^{in}+\delta_k) \ ,\\
\mathrm{G}^{in}_{yy} &= \varGamma_0^{in} + 2\varGamma_1^{in} + \varLambda_0^{in}\cos 4(\varOmega_0^{in}+\delta_k) - 4\varLambda_1^{in}\cos 2(\varOmega_1^{in}+\delta_k) \ ,\\
\mathrm{G}^{in}_{ys} &= -\varLambda_0^{in}\sin 4(\varOmega_0^{in}+\delta_k) + 2\varLambda_1^{in}\sin 2(\varOmega_1^{in}+\delta_k) \ ,\\
\mathrm{G}^{in}_{ss} &= \varGamma_0^{in} - \varLambda_0^{in}\cos 4(\varOmega_0^{in}+\delta_k) \ ,
\end{aligned}
\tag{41}
$$

and

$$
\begin{aligned}
\mathrm{G}^{out}_{qq} &= \varGamma^{out} + \varLambda^{out}\cos 2(\varOmega^{out}-\delta_k) \ ,\\
\mathrm{G}^{out}_{qr} &= \varLambda^{out}\sin 2(\varOmega^{out}-\delta_k) \ ,\\
\mathrm{G}^{out}_{rr} &= \varGamma^{out} - \varLambda^{out}\cos 2(\varOmega^{out}-\delta_k) \ ,
\end{aligned}
\tag{42}
$$

and those of $\{g^{in}\}$:

$$
\begin{aligned}
\mathrm{g}^{in}_{x} &= \gamma^{in} + \lambda^{in}\cos 2(\omega^{in}+\delta_k) \ ,\\
\mathrm{g}^{in}_{s} &= \lambda^{in}\sin 2(\omega^{in}+\delta_k) \ ,\\
\mathrm{g}^{in}_{y} &= \gamma^{in} - \lambda^{in}\cos 2(\omega^{in}+\delta_k) \ .
\end{aligned}
\tag{43}
$$

In Eqs. (41)-(43) $\varGamma_0^{in}$, $\varGamma_1^{in}$, $\varLambda_0^{in}$, $\varLambda_1^{in}$, $\varOmega_0^{in}$ and $\varOmega_1^{in}$ are the polar parameters of the in-plane strength matrix of the lamina; $\varGamma^{out}$, $\varLambda^{out}$ and $\varOmega^{out}$ are the polar parameters of the out-of-plane strength matrix of the lamina; γ^{in}, λ^{in} and ω^{in} are the polar parameters of the in-plane strength vector of the lamina. All of these parameters solely depend upon the ply material strength properties, see [9, 36].

In the framework of the polar analysis of the laminate strength it is useful to homogenise the units of matrices $[G_A]$, $[G_B]$, $[G_D]$, $[G_H]$ and vectors $\{g_A\}$ and $\{g_D\}$ to those of the ply counterparts as follows:

$$
\begin{aligned}
&[G_A^*] = \frac{1}{h}\,[G_A]\,, \quad [G_B^*] = \frac{2}{h^2}\,[G_B]\,, \quad [G_D^*] = \frac{12}{h^3}\,[G_D]\,, \\
&[G_H^*] = \frac{1}{h}\,[G_H]\,, \quad \{g_A^*\} = \frac{1}{h}\,\{g_A\}\,, \quad \{g_D^*\} = \frac{2}{h^2}\,\{g_D\}\,.
\end{aligned}
\tag{44}
$$

The analogy between Eqs. (40) and (14) clearly appears: matrices $[G_A]$, $[G_B]$, $[G_D]$, $[G_H]$ represent the *strength counterpart* of the laminate stiffness matrices $[A]$, $[B]$, $[D]$ and $[H]$. In the case of the polar analysis of the laminate strength the new terms are the membrane and bending strength vectors which are related to the linear part of the laminate failure index. The components of the previous matrices and vectors can be expressed in terms of their characteristic polar parameters. Moreover the laminate strength polar parameters can be easily related to their lamina counterpart. In fact, through the polar formalism, it is easy to separate the contributions of both the geometric parameters of the stack (position, orientations) from the lamina strength properties to the global laminate strength behaviour. In particular, the homogenised membrane, membrane/bending coupling and bending strength matrices behave like fourth-rank elasticity like tensor, the homogenised shear matrix behave like a “special” second-rank symmetric tensor(here the term “special” means that the anisotropic part of this tensor can be got by considering the opposite of the orientation angle for each ply), while the membrane and bending strength vectors behave like a second-rank symmetric tensor. For a laminated plate composed of identical plies, the following relationships apply.

- Polar parameters of $[G_A^*]$:

$$
\begin{aligned}
&\mathit{\Gamma}_0^{G_A^*} = \mathit{\Gamma}_0^{in}\ , \\
&\mathit{\Gamma}_1^{G_A^*} = \mathit{\Gamma}_1^{in}\ , \\
&\mathit{\Lambda}_0^{G_A^*} e^{i4\mathit{\Omega}_0^{G_A^*}} = \frac{1}{n}\mathit{\Lambda}_0^{in} e^{i4\mathit{\Omega}_0^{in}} \sum_{k=1}^{n} e^{i4\delta_k}\ , \\
&\mathit{\Lambda}_1^{G_A^*} e^{i2\mathit{\Omega}_1^{G_A^*}} = \frac{1}{n}\mathit{\Lambda}_1^{in} e^{i2\mathit{\Omega}_1^{in}} \sum_{k=1}^{n} e^{i2\delta_k}\ .
\end{aligned}
\tag{45}
$$

- Polar parameters of $[G_B^*]$:

$$\begin{aligned}
&\mathit{\Gamma}_0^{G_B^*} = 0 \ , \\
&\mathit{\Gamma}_1^{G_B^*} = 0 \ , \\
&\mathit{\Lambda}_0^{G_B^*} e^{i4\mathit{\Omega}_0^{G_B^*}} = \frac{1}{n^2}\mathit{\Lambda}_0^{in} e^{i4\mathit{\Omega}_1^{in}} \sum_{k=1}^{n} b_k e^{i4\delta_k} \ , \\
&\mathit{\Lambda}_1^{G_B^*} e^{i2\mathit{\Omega}_1^{G_B^*}} = \frac{1}{n^2}\mathit{\Lambda}_1^{in} e^{i2\mathit{\Omega}_1^{in}} \sum_{k=1}^{n} b_k e^{i2\delta_k} \ .
\end{aligned} \tag{46}$$

- Polar parameters of $[G_D^*]$:

$$\begin{aligned}
&\mathit{\Gamma}_0^{G_D^*} = \mathit{\Gamma}_0^{in} \ , \\
&\mathit{\Gamma}_1^{G_D^*} = \mathit{\Gamma}_1^{in} \ , \\
&\mathit{\Lambda}_0^{G_D^*} e^{i4\mathit{\Omega}_1^{G_D^*}} = \frac{1}{n^3}\mathit{\Lambda}_0^{in} e^{i4\mathit{\Omega}_1^{in}} \sum_{k=1}^{n} d_k e^{i4\delta_k} \ , \\
&\mathit{\Lambda}_1^{G_D^*} e^{i2\mathit{\Omega}_1^{G_D^*}} = \frac{1}{n^3}\mathit{\Lambda}_1^{in} e^{i2\mathit{\Omega}_1^{in}} \sum_{k=1}^{n} d_k e^{i2\delta_k} \ .
\end{aligned} \tag{47}$$

- Polar parameters of $[G_H^*]$:

$$\begin{aligned}
&\mathit{\Gamma}^{G_H^*} = \mathit{\Gamma}^{out} \ , \\
&\mathit{\Lambda}^{G_H^*} e^{i2\mathit{\Omega}^{G_H^*}} = \frac{1}{n}\mathit{\Lambda}^{out} e^{i2\mathit{\Omega}^{out}} \sum_{k=1}^{n} e^{-i2\delta_k} \ .
\end{aligned} \tag{48}$$

- Polar parameters of $\{g_A^*\}$:

$$\begin{aligned}
&\gamma^{g_A^*} = \gamma^{in} \ , \\
&\lambda^{g_A^*} e^{i2\omega^{g_A^*}} = \frac{1}{n}\lambda^{in} e^{i2\omega^{in}} \sum_{k=1}^{n} e^{i2\delta_k} \ .
\end{aligned} \tag{49}$$

- Polar parameters of $\{g_D^*\}$:

$$\begin{aligned}
&\gamma^{g_D^*} = 0 \ , \\
&\lambda^{g_D^*} e^{i2\omega^{g_D^*}} = \frac{1}{n^2}\lambda^{in} e^{i2\omega^{in}} \sum_{k=1}^{n} b_k e^{i2\delta_k} \ .
\end{aligned} \tag{50}$$

From Eqs. (45)-(50) it seems that, at the macro-scale, the strength of the laminate is governed by 27 polar parameters: six for each one of matrices $[G_A^*]$, $[G_B^*]$ and $[G_D^*]$; three for shear strength matrix $[G^*_H]$ and three for each one of vectors $\{g^*_A\}$ and $\{g^*_D\}$. However some simplifications arise:

- the isotropic moduli of $[G^*_A]$, $[G^*_D]$ and $[G^*_H]$ are identical to the isotropic moduli of the strength matrices of the layer;
- the isotropic moduli of $[G^*_B]$ are null;
- the spherical part of $\{g^*_A\}$ is identical to the spherical part of the strength vector of the layer;
- the spherical part of $\{g^*_D\}$ is null;
- the deviatoric modulus and the polar angle of matrix $[G^*_H]$ can be expressed as a linear combination of the anisotropic polar modulus $\Lambda_1^{G^*_A}$ and the related polar angle $\Omega_1^{G^*_A}$ of matrix $[G^*_A]$ as follows:

$$\Lambda^{G^*_H} e^{i2\Omega^{G^*_H}} = \Lambda_1^{G^*_A} \frac{\Lambda^{out}}{\Lambda_1^{in}} e^{i2(\Omega^{out}+\Omega_1^{in}-\Omega_1^{G^*_A})} \ ; \tag{51}$$

- the deviatoric modulus and the polar angle of vector $\{g^*_A\}$ can be expressed as a linear combination of the anisotropic polar modulus $\Lambda_1^{G^*_A}$ and the related polar angle $\Omega_1^{G^*_A}$ of matrix $[G^*_A]$ as follows:

$$\lambda^{g^*_A} e^{i2\omega^{g^*_A}} = \Lambda_1^{G^*_A} \frac{\lambda^{in}}{\Lambda_1^{in}} e^{i2(\Omega_1^{G^*_A}+\omega^{in}-\Omega_1^{in})} \ ; \tag{52}$$

- the deviatoric modulus and the polar angle of vector $\{g^*_D\}$ can be expressed as a linear combination of the anisotropic polar modulus $\Lambda_1^{G^*_B}$ and the related polar angle $\Omega_1^{G^*_B}$ of matrix $[G^*_B]$ as follows:

$$\lambda^{g^*_D} e^{i2\omega^{g^*_D}} = \Lambda_1^{G^*_B} \frac{\lambda^{in}}{\Lambda_1^{in}} e^{i2(\Omega_1^{G^*_B}+\omega^{in}-\Omega_1^{in})} \ . \tag{53}$$

The details of the proof leading to Eqs. (51)-(53) are given in Appendices Appendix A and Appendix B. The previous considerations have a direct impact on the number of independent polar parameters describing the overall strength of the laminate which reduces to only 12. In particular the independent polar parameters are the anisotropic polar moduli and the polar angles of matrices $[G_A^*]$, $[G_B^*]$ and $[G_D^*]$. This means that, as in the case of the polar analysis of the laminate stiffness matrices, the number of polar parameters to be designed (concerning the laminate strength) remains unchanged when passing from the CLT to the FSDT.

5.2 The relationship between laminate stiffness and strength

The remark made at the end of the previous section can be extended when comparing the matrices governing the behaviour of the laminate in terms of stiffness and strength, respectively. In particular, further fundamental results can be deduced when comparing Eqs. (45)-(47) to Eqs. (20)-(22). In fact the terms $\sum_{k=1}^{n} \alpha e^{im\delta_k}$ (with $\alpha = 1, b_k, d_k$ and $m = 2, 4$) can be used to relate the components of the laminate strength and stiffness matrices, respectively. The analytical formulae expressing the relationship among the stiffness and strength polar parameters of the laminate are reported here below. The proof is omitted for sake of brevity. Nevertheless, these relationships can be easily got by utilising an analogous procedure to that detailed in Appendix Appendix A and Appendix B adapted to the case of four-rank plane tensors. In particular, the anisotropic moduli as well as the polar angles of matrices $[G_A^*]$, $[G_B^*]$ and $[G_D^*]$ can be related to those of the laminate stiffness matrices $[A^*]$, $[B^*]$ and $[D^*]$ as follows:

$$\Lambda_0^{G_A^*} e^{i4\Omega_0^{G_A^*}} = R_0^{A^*} \frac{\Lambda_0^{in}}{R_0} e^{i4(\Phi_0^{A^*} + \Omega_0^{in} - \Phi_0)} \ , \tag{54}$$

$$\Lambda_1^{G_A^*} e^{i2\Omega_1^{G_A^*}} = R_1^{A^*} \frac{\Lambda_1^{in}}{R_1} e^{i2(\Phi_1^{A^*} + \Omega_1^{in} - \Phi_1)} \ , \tag{55}$$

$$\Lambda_0^{G_B^*} e^{i4\Omega_0^{G_B^*}} = R_0^{B^*} \frac{\Lambda_0^{in}}{R_0} e^{i4(\Phi_0^{B^*} + \Omega_0^{in} - \Phi_0)} \ , \tag{56}$$

$$\Lambda_1^{G_B^*} e^{i2\Omega_1^{G_B^*}} = R_1^{B^*} \frac{\Lambda_1^{in}}{R_1} e^{i2(\Phi_1^{B^*} + \Omega_1^{in} - \Phi_1)} \ , \tag{57}$$

$$\Lambda_0^{G_D^*} e^{i4\Omega_0^{G_D^*}} = R_0^{D^*} \frac{\Lambda_0^{in}}{R_0} e^{i4(\Phi_0^{D^*} + \Omega_0^{in} - \Phi_0)} \ , \tag{58}$$

$$\Lambda_1^{G^*_D} e^{i2\Omega_1^{G^*_D}} = R_1^{D^*} \frac{\Lambda_1^{in}}{R_1} e^{i2(\Phi_1^{D^*} + \Omega_1^{in} - \Phi_1)} \ . \tag{59}$$

Eqs. (54)-(59) imply that (when the material of the elementary ply is fixed *a priori*) the overall mechanical response of the laminate, in terms of both stiffness and strength, depends only on the anisotropic polar moduli and the polar angles of matrices [A*], [B*] and [D*] or, equivalently, on those of matrices $[\mathrm{G}^*_{\mathrm{A}}]$, $[\mathrm{G}^*_{\mathrm{B}}]$ and $[\mathrm{G}^*_{\mathrm{D}}]$. More precisely this result show that, at the macroscopic scale (i.e. that of the laminate) stiffness and strength are strictly related and dependent: optimising the stiffness behaviour of a laminate implicitly implies an optimisation of its "average strength" and vice-versa. Last but not least, the number of polar parameters to be designed remains unchanged when passing from the context of CLT to that of FSDT: the designer can act (through a variation of geometric parameters such as layers orientations and positions) only on the anisotropic polar moduli and polar angles of the membrane, membrane/bending coupling and bending stiffness (or strength) matrices, all of the other quantities being directly linked to them.

6 Conclusions

The present study represents a generalisation of the unified approach proposed in [19]: here tensorial laminate-level failure criteria are expressed in the framework of the FSDT in order to catch the influence of the out-of-plane shear stresses on the laminate failure mechanisms. To this purpose, the most common failure criteria of Tsai-Hill, Hoffman, Tsai-Wu and Zhang-Evans are considered and reformulated at the laminate level.

In this work the polar method has been utilised to represent both strength matrices and strength vectors of the laminate (these latter are related to the linear part of the laminate failure index). In particular, the homogenised membrane, membrane/bending coupling and bending strength matrices behave like fourth-rank elasticity-like tensor, the homogenised shear matrix behaves like a "special" second-rank symmetric tensor (here the term "special" means that the anisotropic part of this tensor can be got by considering the opposite of the orientation angle for each ply), while the membrane and bending strength vectors behave like second-rank symmetric tensors.

The polar analysis of the laminate strength behaviour in the FSDT framework lets arise some interesting facts which constitute just as many theoretical results. Firstly the number of independent laminate strength invariants reduces to only 12. In particular, it has been proven that the laminate strength can be completely described through the anisotropic polar moduli and the polar angles of membrane, membrane/bending coupling and bending strength matrices. This means that, as in the case of the polar analysis of the laminate stiffness matrices, the number of polar parameters to be designed (concerning the laminate strength) remains unchanged when passing from the CLT to the FSDT.

Finally by considering the same approach utilised for the polar analysis of the FSDT [4, 20] a further important theoretical result is proven: the existence of a set of analytical relationships between the laminate strength and stiffness invariants implying that (when the material of the elementary ply is fixed *a priori*) the overall mechanical response of the laminate, in terms of both stiffness and strength, depends only on the anisotropic polar moduli and the polar angles of stiffness matrices or, equivalently, on their strength counterparts. This result shows that, at the macroscopic scale (i.e. that of the laminate) stiffness and strength are strictly related and interdependent: optimising the stiffness behaviour of a laminate implicitly implies an optimisation of its "average strength" and vice-versa.

Appendix A The link between the polar parameters of $[G^*_H]$ and those of $[G^*_A]$

In order to analytically derive the link between the deviatoric part of matrix $[G^*_H]$ and the second anisotropic polar modulus $\Lambda_1^{G^*_A}$ and the related polar angle $\Omega_1^{G^*_A}$ of matrix $[G^*_A]$, let us consider the expression of the quantities $\sum_{k=1}^{n} e^{-\mathrm{i}2\delta_k}$ appearing in Eq. (48). These quantities actually depend upon the polar parameters of the membrane strength matrix of the laminate. A quick glance to Eqs. (45) suffices to determine their expression. To derive these relationships let us consider the following property of complex numbers:

$$\overline{\alpha z+\beta w}=\alpha\bar{z}+\beta\bar{w}\ ;\ \text{with } z,w\in\mathbb{C} \text{ and } \alpha,\beta\in\mathbb{R}\ , \tag{A.0}$$

where $\bar{z}$ is the complex conjugate of z. By using property (A.0) and considering Eq. (45) we have:

$$\sum_{k=1}^{n} e^{-\mathrm{i}2\delta_k}=\sum_{k=1}^{n}\overline{e^{\mathrm{i}2\delta_k}}=\overline{\sum_{k=1}^{n} e^{\mathrm{i}2\delta_k}}=n\frac{\Lambda_1^{G_A^*}}{\Lambda_1^{in}}\overline{e^{\mathrm{i}2(\Omega_1^{G_A^*}-\Omega_1^{in})}}=n\frac{\Lambda_1^{G_A^*}}{\Lambda_1^{in}}e^{\mathrm{i}2(\Omega_1^{in}-\Omega_1^{G_A^*})}\ , \tag{A.1}$$

Finally, by substituting Eq. (A.1) into Eq. (48) (and after some standard passages) it is possible to obtain the desired result:

$$\Lambda^{G_H^*}e^{\mathrm{i}2\Omega^{G_H^*}}=\Lambda_1^{G_A^*}\frac{\Lambda^{out}}{\Lambda_1^{in}}e^{\mathrm{i}2(\Omega^{out}+\Omega_1^{in}-\Omega_1^{G_A^*})}\ . \tag{A.2}$$

Appendix B The link between the polar parameters of $\{g_A^*\}$ and $\{g_D^*\}$ and those of $[G_A^*]$ and $[G_B^*]$

Let us consider the expression of the quantities $\sum_{k=1}^{n} e^{\mathrm{i}2\delta_k}$ and $\sum_{k=1}^{n} b_k e^{\mathrm{i}2\delta_k}$ appearing in Eqs. (49)-(50). These quantities actually depend upon the polar parameters of the membrane and membrane/bending coupling strength matrices of the laminate. A quick glance to Eqs. (45) and (46) suffices to determine their expression. Indeed, from Eq. (45) we have:

$$\sum_{k=1}^{n} e^{\mathrm{i}2\delta_k}=\frac{n\Lambda_1^{G_A^*}e^{\mathrm{i}2\Omega_1^{G_A^*}}}{\Lambda_1^{in}e^{\mathrm{i}2\Omega_1^{in}}}=n\frac{\Lambda_1^{G_A^*}}{\Lambda_1^{in}}e^{\mathrm{i}2(\Omega_1^{G_A^*}-\Omega_1^{in})}\ , \tag{B.0}$$

while from Eq. (46) we obtain:

$$\sum_{k=1}^{n} b_k e^{\mathrm{i}2\delta_k}=\frac{n^2\Lambda_1^{G_B^*}e^{\mathrm{i}2\Omega_1^{G_B^*}}}{\Lambda_1^{in}e^{\mathrm{i}2\Omega_1^{in}}}=n^2\frac{\Lambda_1^{G_B^*}}{\Lambda_1^{in}}e^{\mathrm{i}2(\Omega_1^{G_B^*}-\Omega_1^{in})}\ . \tag{B.1}$$

By substituting Eqs. (B.0) and (B.1) into Eqs. (49) and (50), respectively (and after some standard passages) we can obtain the desired result:

$$\lambda^{g_A^*} e^{i2\omega^{g_A^*}} = \Lambda_1^{G_A^*} \frac{\lambda^{in}}{\Lambda_1^{in}} e^{i2(\Omega_1^{G_A^*} + \omega^{in} - \Omega_1^{in})} , \tag{B.2}$$

and:

$$\lambda^{g_D^*} e^{i2\omega^{g_D^*}} = \Lambda_1^{G_B^*} \frac{\lambda^{in}}{\Lambda_1^{in}} e^{i2(\Omega_1^{G_B^*} + \omega^{in} - \Omega_1^{in})} . \tag{B.3}$$

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