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Convex formulation of confidence level optimization of DG affine reactive power controllers in distribution grids

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Abstract

Volatile productions and consumptions generate a stochastic behavior of distribution grids and make its supervision difficult to achieve. Usually, the Distributed Generators reactive powers are adjusted to perform decentralized voltage control. Industrial controllers are generally equipped with a local affine feedback law, which settings are tuned at early stage using local data. A centralized and more efficient tuning method should aim to maximize the probability that all the node voltages of distribution grids remain within prescribed bounds. When the characteristics of the stochastic power forecasts are known, the centralized algorithm allows to update the settings on a regular time basis. However, the method requires to solve stochastic optimization problem. Assuming that stochastic variables have Gaussian distributions, a procedure is given which guarantees the convergence of the stochastic optimization. Convex problems drastically reduce the difficulty and the computational time required to reach the global minimum, compared to nonconvex optimal power flow problems. The linear controllers with optimized parameters are compared to traditional control laws using simulations of a real distribution grid model. The results show that the algorithm is reliable and moreover fast enough. Hence, the proposed method can be used to update periodically the control parameters.

Keywords: Confidence level optimization; control tuning; distribution network; stochastic power flow; convex optimization problem; fractional programming problem

1. Introduction

When a distribution grid contains only loads, the voltage decreases along the lines. However, when power sources such as Distributed Generators (DG) with renewable energies are inserted in a distribution network, reverse active and reactive power flows appear which in turn may generate voltage overshoots [1]. The risk of voltage overshoot can be limited by controlling jointly the On load Tap Changer (OLTC) of the HV/MV transformer and the DGs reactive powers. However, the consumption and production powers are stochastic and vary during the course of the day. Hence, a traditional decentralized droop-based control architecture with unchanging parameters generates sub-optimal control efforts.

A first range of solutions is real-time controllers which are able to anticipate or cope with changing behaviors, such as Model Predictive Controllers (e.g. [2]), adaptive droop controllers (e.g.[3]), or Coordinated Voltage Control (e.g. [4]). In practice, centralized and high sampling rate controllers are not suitable for big-size grids when they embed limited communication and actuation technologies. A reasonable alternative is to keep a decentralized structure with affine control laws and to optimize droop parameters on a longer time span, this solution agrees with the industrial communication system capacities of the distribution grids. This method allows to reduce DG efforts

thanks to a global optimization and the use of forecasting.

Among the different optimization strategies, stochastic optimization has many interests: first, it can take uncertainties into account. Then, it can optimize expectations, variances, confidence levels and risks. Algorithms using confidence levels take advantage of the stochastic nature of the variables and are able to provide a likelihood of violation [5, 6], this is a major benefit over robust methods which use only uncertainty bounds [7, 8]. However, these algorithms only consider chance constraints with a fixed confidence level p , whether applied to droop gain synthesis (e.g. [4]), real-time control design [9, 10] or Unit Commitment problems (see e.g. [11, 12, 13]). When p is high, the system is more secure, albeit at a higher economic cost. Very little is said in the literature on how to obtain the best rate of service (called a *max-p problem*), that is to say find the set of decentralized controllers parameters which maximizes the probability that all node voltages remain within the prescribed bounds [14, 5]. The adjustment is mainly done according to experience or conventions [15, 16].

In order to update droop gains within 15-60 minutes, it is necessary to select an optimization algorithm that converge quickly to a global optimum. This calls for the use of interior point methods which handle efficiently convex problems [17] and discards Monte-Carlo methods for the optimization of large-scale grids

Nomenclature

$\alpha_{ii}, \beta_{ii}, Q_i^0$	Control parameters of the affine law of the DG at node i	$\mu_{\tilde{X}}, \sigma_{\tilde{X}}$	Mean and standard deviation of the stochastic variable \tilde{X}
η, κ	Confidence levels	$\tilde{\mathbf{I}}$	Vector of currents entering or exiting nodes ($\tilde{\mathbf{I}} = [\tilde{I}_1, \dots, \tilde{I}_n]^T$)
$\tilde{\mathbf{S}}$	Vector of complex powers ($\tilde{\mathbf{S}} = \tilde{\mathbf{P}} + j\tilde{\mathbf{Q}}$)	$\tilde{\mathbf{V}}$	Vector of voltages, nodes 1 to n : $\tilde{\mathbf{V}} = [\tilde{V}_1, \dots, \tilde{V}_n]^T$
$\tilde{\mathbf{P}}^l$	Vector of consumption active powers ($\tilde{\mathbf{P}}^l = [\tilde{P}_1^l, \dots, \tilde{P}_n^l]^T$)	\tilde{I}_0	Currents entering or exiting slack bus 0
$\tilde{\mathbf{P}}^p$	Vector of production reactive powers ($\tilde{\mathbf{P}}^p = [\tilde{P}_1^p, \dots, \tilde{P}_n^p]^T$)	\tilde{V}_0	OLTC node voltage
$\tilde{\mathbf{P}}$	Vector of active powers ($\tilde{\mathbf{P}} = [\tilde{P}_1, \dots, \tilde{P}_n]^T$)	\tilde{V}_0^{err}	OLTC node voltage error
$\tilde{\mathbf{Q}}^l$	Vector of consumption reactive powers ($\tilde{\mathbf{Q}}^l = [\tilde{Q}_1^l, \dots, \tilde{Q}_n^l]^T$)	$n + 1, G$	Number of nodes, number of DGs
$\tilde{\mathbf{Q}}^p$	Vector of production reactive powers ($\tilde{\mathbf{Q}}^p = [\tilde{Q}_1^p, \dots, \tilde{Q}_n^p]^T$)	P_i^N	DG nominal power at node i
$\tilde{\mathbf{Q}}$	Vector of reactive powers ($\tilde{\mathbf{Q}} = [\tilde{Q}_1, \dots, \tilde{Q}_n]^T$)	V_0^{ref}	OLTC node voltage reference
\mathbf{Y}	Admittance matrix	w_i	Objective function weighting factors
\mathbf{Z}	Impedance matrix, $\mathbf{Z} = \mathbf{Y}^{-1}$	$x \mapsto \Phi(x)$	Standard Gaussian cumulative distribution function
$\mathbf{1}$	Vector of 1s, dimension n	$x \mapsto \phi(x)$	Standard Gaussian probabilistic distribution function
		DG	Distributed Generator
		OLTC	On Load Tap Changer

[18, 19] which bear a high computational burden. However, in general, chance constrained or max-p problems are not convex [20]. The application to chance constrained droop control synthesis in a power grid [4] has been handled with linear approximations. [5] considered piecewise affine laws (e.g. a dead band) that are sometimes applied to DG control. It was shown that the node voltages could be considered as combinations of piecewise Gaussian variables, but the max-p problem was not shown explicitly to be convex.

This paper proposes a generic stochastic algorithm which optimizes the DGs droop control parameters. The objective function and constraints can be defined as a weighted sum of squares of expectations, variances, confidence levels or risk levels of any voltages or DG powers. Moreover, it is shown that this algorithm uses convex subproblems and converges very quickly. Section II shows that, under decentralized affine control, the grid voltages can be approximated accurately by a linear mapping of the control parameters and the stochastic powers or consumptions, using the quasi-static linear grid model of [21]. In section III, the max-p problem is recast into a generalized fractional programming problem, that is to say find the minimax of several fractions. Such a problem can be solved by a Newton-like algorithm called Dinkelbach's procedure [22, 23], and considers subproblems that will be shown to be convex. A simulation of a real distribution grid under optimal control is given in section IV. This example will be used to assess numerically the convexity of the optimization problem and the relevance of the

optimization procedure.

2. Simplified stochastic model of a distribution grid

2.1. Linear stochastic power flow model under linear control

In the paper, a subscript $()_i$ will indicate the value of the corresponding variable at node i , an exponent will refer to the membership of the variable to a specific set which will be defined in the text. A tilde (\sim) placed on top of a symbol indicates that the corresponding variable is stochastic. Usually, voltages and powers in a distribution grid with $n + 1$ nodes are obtained from a AC power flow (1, 2), which accounts for the power balance at every node i .

$$\tilde{P}_i = V_i \sum_{j=0}^n \tilde{V}_j Y_{ij} \cos(\Delta_i - \Delta_j - \Theta_{ij}) \quad (1)$$

$$\tilde{Q}_i = V_i \sum_{j=0}^n \tilde{V}_j Y_{ij} \sin(\Delta_i - \Delta_j - \Theta_{ij}) \quad (2)$$

where $\tilde{P}_i, \tilde{Q}_i, \tilde{V}_i, \Delta_i$ are injected active and reactive powers, voltage and angle at node i , Y_{ij}, Θ_{ij} are admittance matrix parameters. However, these models are nonlinear and their use in an optimization algorithm is computationally prohibitive or no convergence can be guaranteed. Instead, it is possible to use quasi-static linear models which provide an estimation of the nodes voltages (3) ($\tilde{\mathbf{V}}$) as a function of load/production reactive ($\tilde{\mathbf{P}}$) and reactive ($\tilde{\mathbf{Q}}$) powers with a remarkable accuracy.

This linear model has been originally published in [21] and has been extended in [24]. Paper [24] proposes a more general model which considers networks with shunt elements, several slack buses, and displays a larger validity domain. The practical performances and validity domain of this model have been evaluated using an industrial distribution grid in [25].

$$\tilde{\mathbf{V}} = \mathbf{A}\tilde{\mathbf{P}} + \mathbf{B}\tilde{\mathbf{Q}} + \mathbf{1}\tilde{V}_0 \quad (3)$$

where the reference angle is the voltage of the bus 0 (V_0) which refers to the secondary side of the HV/MV distribution transformer, and is controlled by an On Load Tap Changer (OLTC). The theoretical upper bound of the voltage estimation error ϵ_i (4) at the node i is given by:

$$|\epsilon_i| \leq \frac{4}{V_0^3} \|Z_i\| \|\mathbf{Z}\|^* \|\tilde{\mathbf{S}}\| \quad (4)$$

where $\mathbf{Z} = \mathbf{Y}^{-1}$ is the inverse of the admittance matrix \mathbf{Y} and Z_i its i^{th} row. The norm $\|\cdot\|$ refers to vector norm and the symbol $\|\cdot\|^*$ refers to the norm defined in (5).

$$\|\mathbf{Z}\|^* = \max_h \|e_h^T \mathbf{Z}\| \quad (5)$$

where e_h^T is the h -th vector of the canonical base.

Simulations of real distribution networks have shown that the real error is lower than 0.001 pu for 86% of configurations of real networks. The theoretical error is below 0.01 pu [25]. A simple estimation of current lines is obtained by the admittance matrix and Ohm's law (6).

$$\tilde{\mathbf{I}} = \mathbf{A}'\tilde{\mathbf{P}} + \mathbf{B}'\tilde{\mathbf{Q}} + \tilde{\mathbf{I}}_0 \quad (6)$$

The linear model in (4) includes stochastic active $\tilde{\mathbf{P}}$ and reactive $\tilde{\mathbf{Q}}$ powers which can be split respectively in consumptions $\tilde{\mathbf{P}}^l$ and $\tilde{\mathbf{Q}}^l$ and productions $\tilde{\mathbf{P}}^g$ and $\tilde{\mathbf{Q}}^g$ as described in (7) and (8). The nature of the network itself (lines, equipment, controllers, etc.) is deterministic. However, the forecasting of the system state can take uncertainties errors into account. For example, Fig. 1 provides the most probable forecasts of a consumption for a 24 hour span and the corresponding uncertainties.

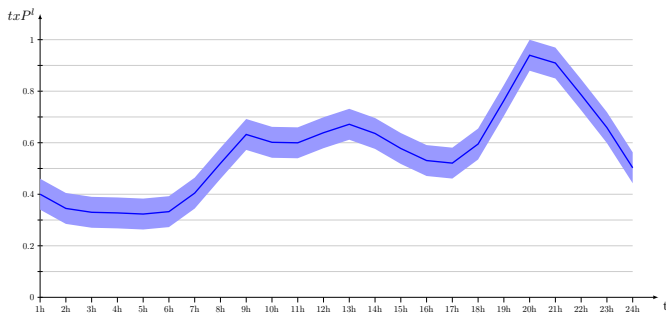


Figure 1. Stochastic distribution of forecasting error of PV production

The power terms are split into consumption and production terms in (7, 8) and the forecasting errors are provided by historical data.

$$\tilde{\mathbf{P}} = \tilde{\mathbf{P}}^l + \tilde{\mathbf{P}}^g \quad (7)$$

$$\tilde{\mathbf{Q}} = \tilde{\mathbf{Q}}^l + \tilde{\mathbf{Q}}^g \quad (8)$$

An example of the distribution of the power forecasting error for a PV production is given in Fig.2, which can be clearly approximated by a Gaussian distribution. In the literature, similar PDF are found in [26, 27] and in Fig. 3.

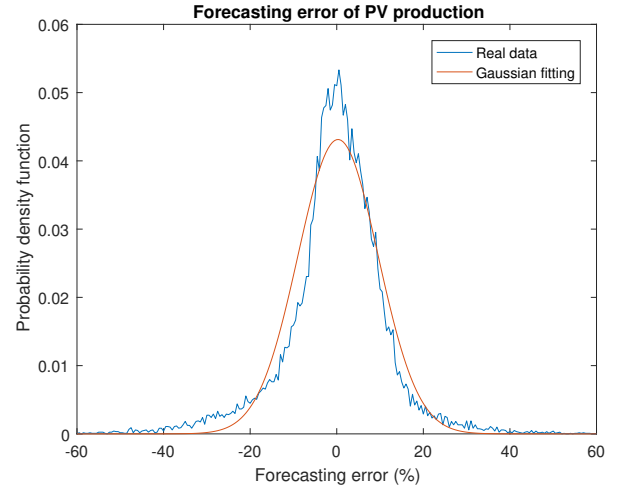


Figure 2. Stochastic distribution of forecasting error of PV production

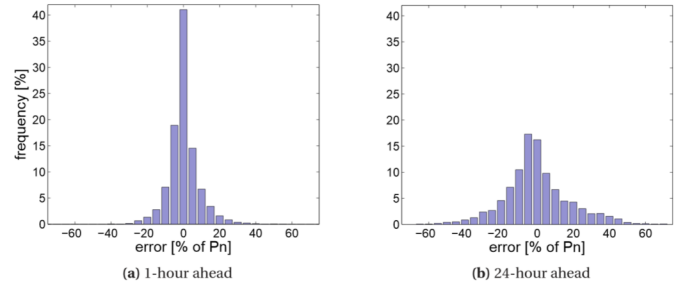


Figure 3. Error of wind power forecasting [27]

Fig.3 shows two examples of the short-term (e.g. daily) wind power forecasting error probability density functions (PDF), which histograms can be approximated by Gaussian distributions. Likewise, stochastic consumptions and productions powers and OLTC voltage error can be modeled as Gaussian variables [28].

In distribution networks, the actuators dedicated to voltage control are the OLTC and the distributed generators reactive powers. The voltage \tilde{V}_0 is controlled by the OLTC which selects the appropriate tap according to the voltage reference V_0^{ref} which is prescribed by the distribution transformer [2]. The OLTC voltage \tilde{V}_0 is defined by (9):

$$\tilde{V}_0 = V_0^{ref} + \tilde{V}_0^{err} \quad (9)$$

The error voltage comes mainly from quantization (tap selection) and is modeled by the Gaussian variable $\tilde{V}_0^{err} = \mathcal{N}(0, 0.05^2)$ [5].

Contrary to the OLTC, the DGs are scattered all over the distribution grid. The structure of the DG reactive power industrial controllers can be either linear or piecewise linear of local measurements such as the DG node voltage and the DG active power production. The matrix representation of the reactive power control is given by equation (10).

$$\tilde{\mathbf{Q}}^g = \alpha \tilde{\mathbf{V}} + \beta \tilde{\mathbf{P}}^g + \mathbf{Q}^0 \quad (10)$$

The matrix $\alpha \in \mathbb{R}^{n \times n}$ is a diagonal matrix for which the G diagonal terms α_{ii} represent the adjustable slopes Q/V of the reactive power affine controllers of the G nodes which have active generators. All other elements of the matrix α are equal to zero, and zero diagonal elements correspond to nodes for which there is no reactive power control. The matrix $\beta \in \mathbb{R}^{n \times n}$ is defined as the same way and represents the adjustable slopes Q/P . The vector $\mathbf{Q}^0 \in \mathbb{R}^n$ represents the constant part of the affine controllers in the same way, where the G adjustable terms correspond to the G nodes with controlled DGs.

The vectors of adjustable control parameters, which is the decision variables in the optimization problem introduced in section 3, be noted $\mathbf{x}^\alpha \in \mathbb{R}^G$, $\mathbf{x}^\beta \in \mathbb{R}^G$, $\mathbf{x}^Q \in \mathbb{R}^G$, respectively for the slopes and constant parameters of the G affine control laws.

Next, replacing (10) into equation (3) yields:

$$(\mathbb{I} - \mathbf{B}\alpha)\tilde{\mathbf{V}} = \mathbf{A}\tilde{\mathbf{P}} + \mathbf{B}(\beta\tilde{\mathbf{P}}^g + \mathbf{Q}^0 + \tilde{\mathbf{Q}}^1) + \mathbb{1}\tilde{V}_0 \quad (11)$$

Note that in equation (11), $\tilde{\mathbf{P}}^g, \tilde{\mathbf{P}}^1, \tilde{\mathbf{Q}}^1, \tilde{V}_0$ are independent stochastic variables, $\mathbf{Q}^0, \alpha, \beta$ are respectively a vector and a diagonal matrix with each G adjustable deterministic parameters.

If all variables which are fed to the linear model (3) are assumed to be Gaussian and independent, Gaussian nodes voltages are obtained, which means and standard deviations can be determined analytically without any iteration.

2.2. Voltages as linear models of control parameters

Solving equation (11) to obtain $\tilde{\mathbf{V}}$ requires to invert a matrix that will distribute nonlinearly the control parameters α over the stochastic powers $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{Q}}^1$. Such nonlinear coupled terms are difficult to handle easily by classical optimization routines.

Lemma 1. Assume that $\tilde{\mathbf{P}}, \tilde{V}_0, \tilde{\mathbf{Q}}^1$ have Gaussian distributions. If $\|\mathbf{B}\alpha\| \ll 1$ and $\|\mathbf{B}\alpha\mathbf{B}\| \ll 1$ (Schatten 2-norm) then the node voltages exhibit a Gaussian distribution and are linear functions of the control parameters α, β and \mathbf{Q}^0 :

$$\tilde{\mathbf{V}} = \mathbf{A}\tilde{\mathbf{P}} + \mathbf{B}\tilde{\mathbf{Q}}^1 + \mathbb{1}\tilde{V}_0 + \mathbf{B}\alpha\mathbf{A}\tilde{\mathbf{P}} + \mathbf{B}\beta\tilde{\mathbf{P}}^g + \mathbf{B}\alpha\mathbb{1}\tilde{V}_0 + \mathbf{B}\alpha\mathbf{B}\tilde{\mathbf{Q}}^1 + \mathbf{B}\mathbf{Q}^0 \quad (12)$$

Proof. The first assumption allows to use a first order Taylor expansion $(\mathbb{I} - \mathbf{B}\alpha)^{-1} \simeq (\mathbb{I} + \mathbf{B}\alpha)$. Replacing into (11) brings $\tilde{\mathbf{V}} = (\mathbb{I} + \mathbf{B}\alpha)(\mathbf{A}\tilde{\mathbf{P}} + \mathbf{B}\beta\tilde{\mathbf{P}}^g + \mathbf{B}\tilde{\mathbf{Q}}^1 + \mathbf{B}\mathbf{Q}^0 + \mathbb{1}\tilde{V}_0)$. Using the second assumption to neglect the term $\mathbf{B}\alpha\mathbf{B}\mathbf{Q}^0$ and $\mathbf{B}\alpha\mathbf{B}\beta\tilde{\mathbf{P}}^g$ yields equation (12). The expression of $\tilde{\mathbf{V}}$ is clearly a linear combination of the independent stochastic variables $(\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^1, \tilde{V}_0)$ and a linear combination of the control parameters $(\alpha, \beta, \mathbf{Q}^0)$. Note that only G elements of each control vector are nonzero.

It is well known that the sum of independent Gaussian variables is also Gaussian. Hence, assuming that $\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}^1, \tilde{V}_0$ have Gaussian distributions, $\tilde{\mathbf{V}}$ also has a Gaussian distribution. \square

3. Confidence level optimization of decentralized control parameters

3.1. From a confidence level to a generalized fractional programming problem

The goal of the paper is to keep the same standard affine reactive power control structure given equation (10) and find the optimal control parameters $\mathbf{x}^\alpha \in \mathbb{R}^G$, $\mathbf{x}^\beta \in \mathbb{R}^G$, $\mathbf{x}^Q \in \mathbb{R}^G$. The optimization procedure assumes that the distribution of stochastic variables \tilde{V}_0 and $\tilde{\mathbf{P}}$ on an hourly or a daily basis are Gaussian and known a priori. The generic optimization objectives used to tune the control parameters are:

- the minimum probability that voltages remain within prescribed bounds $[V_{min}; V_{max}]$ is maximized ;
- the minimum probability that powers $(\tilde{\mathbf{P}}^g, \tilde{\mathbf{Q}}^g)$ remain within prescribed domain is maximized ;
- the sum of voltage variances is minimized ;
- the sum of squares of DG reactive powers expectations is minimized.

The maximization of the minimum probability that voltages remain within prescribed bounds $[V_{min}; V_{max}]$ is defined in (13):

$$\max_{\mathbf{x}^\alpha, \mathbf{x}^\beta, \mathbf{x}^Q} \min_{i=1 \dots n} \{p_i \mid \mathbb{P}(V_{min} \leq \tilde{V}_i \leq V_{max}) \geq p_i\} \quad (13)$$

where \mathbb{P} is a probability measure. This problem can be written under the standard max-p confidence level optimization program (14), for which p is a joint probability level that corresponds to a service rate:

$$\max_{\mathbf{x}^\alpha, \mathbf{x}^\beta, \mathbf{x}^Q} \{p \mid \mathbb{P}(V_{min} \leq \tilde{V}_i \leq V_{max}) \geq p\} \quad \forall i = 1 \dots n \quad (14)$$

Alternately, maximizing a confidence level is equivalent to minimizing the rate of failure $1 - p$, that is the probability of voltage over or undershoot:

$$\min_{\mathbf{x}^\alpha, \mathbf{x}^\beta, \mathbf{x}^Q} \{1 - p \mid \mathbb{P}(V_{min} \leq \tilde{V}_i \leq V_{max}) \geq p\} \quad \forall i = 1 \dots n \quad (15)$$

The max-p problem is related to chance-constrained problems found in the literature which consider the probabilistic inequality in (15) as a constraint with a specified rate of failure $1 - p$ or confidence level p .

The challenge addressed in this paper is to show the convergence of the optimization problem. Variances and square of expectation are quadratic functions of the parameters and, hence, are convex. However, the convergence of the constrained max-p problem cannot be proved directly as discussed in the next paragraph.

In specific cases, according to the structure of the constraints and the nature of the probability density functions, it is possible to show the convexity of chance-constrained problems or to propose resolution methods which ensure a convergence towards a global optimum [20, 29]. However, there is a challenge to handle max-p problems when random (even Gaussian) and decision (control) variables cannot be decoupled (which

is typically the case here as equation (12) displays bilinear terms). In this case, the max-p problem has to consider a priori non-convex and non-linear multiple objectives, and a specific method has to be developed to solve the problem. In a first step, it will be recalled that a max-p problem can be recast in a generalized fractional programming problem [22], along with a specific application of this transformation to the confidence level optimization problem of equation (12). Then, it will be shown that this problem, while being not convex, can be solved using a Newton-like procedure which converges to the global minimum. This procedure involves the resolution of subprograms, which will be shown to be convex in the specific case considered in the paper, which will allow the use of efficient convex optimization algorithms [17].

Proposition 1. [29] *Let \tilde{c} a random vector with multivariate normal distribution, S a regular region of \mathbb{R}^n (i.e. not empty and bounded), then the problem $\max_{x \in S} \mathbb{P}(\tilde{c}^T x \geq M)$ is equivalent to*

$$\max_{x \in S} \frac{-M + \tilde{c}^T x}{\sqrt{x^T \mathcal{V} x}} \quad (16)$$

where \tilde{c} , \mathcal{V} are respectively the mean and covariance matrix of \tilde{c} .

In the literature, a problem such as (16), which consists of maximizing the ratio of two functions $f(x), g(x)$ of the decision vector is called a *fractional programming problem*. Hence, Proposition 1 states the equivalence between a stochastic optimization problem and a fractional optimization programming problem, which in this case appears to be concave. When it comes to find the maximum of minimum of several fractions, the problem is called a *generalized fractional programming problem* which is more complicated to handle because the denominator is not the same for all fractions, and is in general neither concave nor convex.

$$\lambda = \min_{x \in S} \left\{ \max_{i=1 \dots m} \frac{f_i(x)}{g_i(x)} \right\} \quad (17)$$

where S is a nonempty subset of \mathbb{R}^n , and $f_i, g_i, i = 1 \dots m$ are continuous functions with $g_i(x) > 0, \forall x \in S$.

If the Gaussian characteristics of the power and consumptions distributions are known, a main result of this paper can be stated which recasts the max-p problem into the generalized fractional programming problem (18), which unfortunately turns out to be non-convex. However, it will be seen later that it can be solved by a procedure which embeds convex subproblems.

Proposition 2. *The max-p problem*

$$\max_{x^\alpha, x^\beta, x^Q} \{p \mid \mathbb{P}(\tilde{V}_i \geq V_{min}) \geq p, \forall i = 1 \dots n\}$$

can be recast into the generalized fractional programming problem \mathcal{P} given in (18) or in (19).

$$\mathcal{P} : \max_{x^\alpha, x^\beta, x^Q} \min_{i=1 \dots n} \frac{\mu_{\tilde{V}_i} - V_{min}}{\sigma_{\tilde{V}_i}} \quad (18)$$

$$\mathcal{P} : \min_{x^\alpha, x^\beta, x^Q} \max_{i=1 \dots n} \frac{V_{min} - \mu_{\tilde{V}_i}}{\sigma_{\tilde{V}_i}} \quad (19)$$

Proof.

$$\begin{aligned} & \max_{x^\alpha, x^\beta, x^Q} \{p \mid \forall i = 1 \dots n, \mathbb{P}(\tilde{V}_i \geq V_{min}) \geq p\} \\ \Leftrightarrow & \max_{x^\alpha, x^\beta, x^Q} \left\{ \min_{1 \leq i \leq n} \left\{ \mathbb{P}(\tilde{V}_i \geq V_{min}) \right\} \right\} \\ \Leftrightarrow & \max_{x^\alpha, x^\beta, x^Q} \left\{ \min_{1 \leq i \leq n} \left\{ \mathbb{P}\left(\frac{\tilde{V}_i - \mu_{\tilde{V}_i}}{\sigma_{\tilde{V}_i}} \geq \frac{V_{min} - \mu_{\tilde{V}_i}}{\sigma_{\tilde{V}_i}} \right) \right\} \right\} \\ \Leftrightarrow & \max_{x^\alpha, x^\beta, x^Q} \left\{ \min_{1 \leq i \leq n} \left\{ 1 - \Phi\left(\frac{V_{min} - \mu_{\tilde{V}_i}}{\sigma_{\tilde{V}_i}} \right) \right\} \right\} \\ \Leftrightarrow & \max_{x^\alpha, x^\beta, x^Q} \left\{ \min_{1 \leq i \leq n} \left\{ -\Phi\left(\frac{V_{min} - \mu_{\tilde{V}_i}}{\sigma_{\tilde{V}_i}} \right) \right\} \right\} \\ \Leftrightarrow & \max_{x^\alpha, x^\beta, x^Q} \left\{ \max_{1 \leq i \leq n} \left\{ \Phi\left(\frac{V_{min} - \mu_{\tilde{V}_i}}{\sigma_{\tilde{V}_i}} \right) \right\} \right\} \\ \Leftrightarrow & \max_{x^\alpha, x^\beta, x^Q} \left\{ \max_{1 \leq i \leq n} \left\{ \frac{V_{min} - \mu_{\tilde{V}_i}}{\sigma_{\tilde{V}_i}} \right\} \right\} \\ \Leftrightarrow & \max_{x^\alpha, x^\beta, x^Q} \left\{ \min_{1 \leq i \leq n} \left\{ \frac{\mu_{\tilde{V}_i} - V_{min}}{\sigma_{\tilde{V}_i}} \right\} \right\} \end{aligned}$$

Also, the formulation can use the maximization of the minimum confidence level (18) use the minimization of the maximum risk (19). \square

In the same way, the max-p problem considering the upper bound V_{max} can be recast in a second generalized fractional programming problem (20).

Proposition 3. *The max-p problem*

$$\max_{x^\alpha, x^\beta, x^Q} \{p \mid \mathbb{P}(\tilde{V}_i \leq V_{max}) \geq p, \forall i = 1 \dots n\}$$

can be recast into the generalized fractional programming problem \mathcal{P} given in (20) or (21).

$$\mathcal{P} : \max_{x^\alpha, x^\beta, x^Q} \min_{i=1 \dots n} \frac{V_{max} - \mu_{\tilde{V}_i}}{\sigma_{\tilde{V}_i}} \quad (20)$$

$$\mathcal{P} : \min_{x^\alpha, x^\beta, x^Q} \max_{i=1 \dots n} \frac{\mu_{\tilde{V}_i} - V_{max}}{\sigma_{\tilde{V}_i}} \quad (21)$$

These two generalized fractional programs are intentionally not merged because the full problem does not result into a unique generalized fractional program. Hence, it will be impossible to solve both programs at the same time using convex or fast optimization algorithms. This means that one has to choose whether all voltages -at the same time- should not exceed the upper bound or whether all voltages should remain above the lower bound. In practice, the main problem for the kind of networks which are considered in our study is to avoid overshoots; moreover, the criterion involves the voltage variance, which should yield a probability of occurrence of undershoots smaller than that of overshoots.

3.2. Solving the optimization problem using the generalized Dinkelbach's procedure

There exists a number of methods to solve generalized fractional programming problems, including specific interior point

methods [30], a method of centers [31], or branch and bound algorithms [32]. However, the standard procedure is a Newton-like algorithm called Dinkelbach's procedure which has been generalized to multiple fractions by [22]. When the numerator is convex and the denominator is concave, the problem is more easily tractable as seen later on, even if, regardless of the concavity and the convexity of f_i, g_i , the objective function of a fractional (generalized or not) program is, in general, not concave nor convex. This assumption holds, as will be demonstrated later, for the voltage max-p problem of equation (14). The generalization of the approach designed by Dinkelbach for a single ratio (17), consists of finding a solution of $F(\lambda) = 0$ where

$$F(\lambda) = \min_{x \in S} \left\{ \max_{i=1 \dots n} (f_i(x) - \lambda g_i(x)) \right\} \quad (22)$$

Dinkelbach's procedure relies on the fact that $f_i(x) - \lambda g_i(x)$ is affine in λ and hence $F(\lambda)$ is concave in λ , and calls a Newton method which will be called Procedure 1:

Procedure 1. *Dinkelbach's procedure :*

Step 1: Initialize $\lambda_1 = \max_{i=1 \dots n} (f_i(x^0)/g_i(x^0))$, $x^0 \in X$, $k = 1$

Step 2: Determine an optimal solution x^k of $\mathcal{P}_k : F(\lambda_k) = \min_{x \in S} \left\{ \max_{i=1 \dots n} (f_i(x) - \lambda_k g_i(x)) \right\}$

Step 3: If $F(\lambda_k) = 0$, x^k, λ^k are the optimal values and STOP

Step 4: else $\lambda_{k+1} := \max_{i=1 \dots n} (f(x^k)/g(x^k))$, $k \leftarrow k + 1$; Goto Step 2;

The convergence of the algorithm and the equivalence of finding a root to Dinkelbach's procedure with the generalized fractional problem have been proven in the literature under mild assumptions.

Proposition 4. [22] *If S is compact, the sequence λ_k , if not finite, converges linearly to the optimal value, and each subsequence of x^k converges to an optimal solution of the generalized fractional problem.*

Hence, Dinkelbach's procedure is an efficient algorithm provided that the subproblem \mathcal{P}_k , which has to be considered at each step, can be solved sufficiently fast. The next subsection shows that, in our case, \mathcal{P}_k turns out to be convex.

3.3. Convexity of the subproblems

Proposition 5. *The subproblem $\mathcal{P}_k : F(\lambda_k) = \min_{x^\alpha, x^\beta, x^Q} \left\{ \max_{i=1 \dots n} ((\mu_{\tilde{V}_i} - V_{max}) - \lambda_k \sigma_{\tilde{V}_i}) \right\}$ is convex in x^α, x^Q .*

Proof. $\sigma_{\tilde{V}_i}^2$ is a norm so it is convex. $(\mu_{\tilde{V}_i} - V_{max})$ is an affine function of x^α, x^β, x^Q . As $\lambda_{k+1} := \max_{i=1 \dots n} \frac{\mu_{\tilde{V}_i} - V_{max}}{\sigma_{\tilde{V}_i}} < 0$, $(\mu_{\tilde{V}_i} - V_{max}) - \lambda_k \sigma_{\tilde{V}_i}$ is convex in x^α, x^β, x^Q . The maximum of a set of convex functions is convex itself, which completes the proof. \square

Now, Dinkelbach's procedure can be easily applied, as a convex subproblem has to be solved at each step. Procedure 2 summarizes the algorithm which solves the optimal confidence problem (14):

Procedure 2. *Optimal confidence level procedure :*

Step 1: Compute the voltages means and standard deviations with (12) and initialize Dinkelbach's procedure

Step 2: Follow the steps of Procedure 1 and solve \mathcal{P}_k with a method dedicated to convex problems. A numerical criterion can allow to STOP the algorithm, for example if $|F(\lambda_k)| < \epsilon$, where ϵ is a prescribed value.

In this paper, a standard interior point method, which is a convex optimization algorithm, was used and implemented in Matlab.

To sum up, the complete flowchart is detailed in Fig. 4.

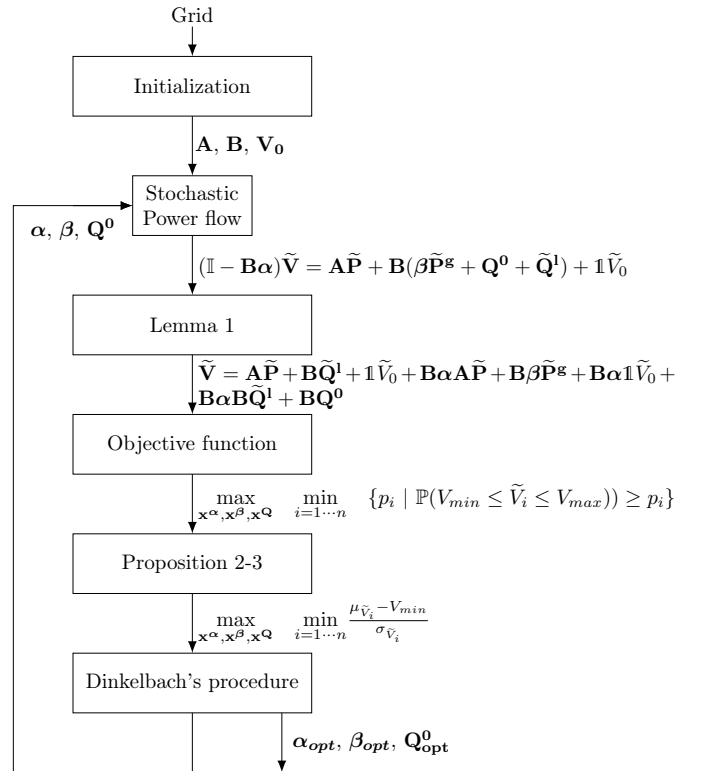


Figure 4. Flowchart of the optimization procedure

4. Practical application to an industrial distribution grid

4.1. Presentation of the grid and optimization problem

The methodology is applied on one of the 27 feeders of a real distribution grid. This 9 km long feeder has 168 nodes, 35 loads (distributed over 26 nodes). The consumption of loads and the production of DGs are respectively 3.1 MW (and 2.1 MVar) and 8.2 MW. These values correspond respectively to 50 % of the maximal consumption and 60 % of the nominal production. The feeder has 7 generators, and the sum of DG nominal powers

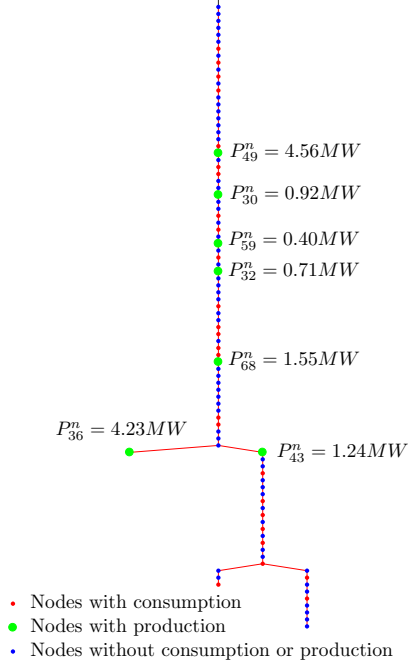


Figure 5. Mixed feeder

is 13.6 MW. The initial network is modified, as two generators receive an extra 3 MW active power (see Fig.5).

The OLTC initial reference voltage is kept at the industrial choice $V_0^{ref} = 1.03$ pu. The goal of the study is to optimize the DGs controllers settings in order to comply with regulatory standards without adding extra devices to the grid or reinforcement.

The means and standard deviations of parameters uncertainties are given in Table 1.

Uncertainty	Mean	Std
Prevision uncertainty of the global grid consumption ($\frac{P_{real}-P_{predict}}{P_{max}}$)	-0.30 %	2.98 %
Consumption spreading uncertainty		50 %
Wind power production uncertainty ($\frac{P_{real}-P_{predict}}{P_{max}}$)	1.71 %	18.10 %
Solar power production uncertainty ($\frac{P_{real}-P_{predict}}{P_{max}}$)	0.30 %	11.56 %
OLTC uncertainty		0.005pu

Table 1. Stochastic input parameters

$$\tilde{\mathbf{Q}}^g = \alpha \tilde{\mathbf{V}} + \beta \tilde{\mathbf{P}}^g + \mathbf{Q}^0 \quad (23)$$

One must find the control parameters $\mathbf{x}^\alpha, \mathbf{x}^\beta, \mathbf{x}^Q$ of the 7 generators which optimize the criterion (24):

$$\max_{\mathbf{x}^\alpha, \mathbf{x}^\beta, \mathbf{x}^Q} \omega_1 \min_{i=0 \dots n} \eta_i + \omega_2 \min_{i=0 \dots n} \kappa_i - \frac{\omega_3}{n} \sum_{i=0}^n Var(\tilde{V}_i) - \frac{\omega_4}{G} \sum_{i=0}^n \frac{E[\tilde{Q}_i^p]^2}{P_i^N} \quad (24)$$

where

- η_i are the confidence level for which every voltage $\tilde{V}_i, i = 1 \dots 168$ lies in the contractual domain, here $[0.95; 1.05]$ pu,
- κ_i are the confidence level that the DG powers of each of the 7 DGs remain in the contractual PQ domains,
- $Var(\tilde{V}_i)$ is the voltage variance at node i ,
- $E[\frac{\tilde{Q}_i^p}{P_i^N}]^2$ is the expectation of the DG control effort at node i when the node hosts a DG; power P_i^N is the DG nominal power.

$\omega_1, \omega_2, \omega_3, \omega_4 > 0$ are weighting factors between the service rates η , the DG specifications λ , voltages variances and control efforts. In this case, the weighting factors used are $\omega = (0.995; 0.995; 1; 0.0005)$. These values are discussed in the subsection 5.3.

Moreover, the confidence levels η_i, δ_i that the constraints respectively on DG powers and line currents are respected should exceed a contractual value. All the mandatory constraints are summed up in equation (25).

$$\eta_i > \eta_s \quad \forall \text{ nodes} \quad (25a)$$

$$\kappa_i > \kappa_s \quad \forall \text{ DGs} \quad (25b)$$

$$\delta_l > \delta_s \quad \forall \text{ lines} \quad (25c)$$

where $\eta_s, \kappa_s, \delta_s$ are contractual values. It is recalled that voltage bounds, current bounds and power bounds are specified either in Grid Codes or in datasheets. The 95% confidence level is not an objective, but only a constraint. It ensures that the probability that the system complies with all regulations or constraints is high enough. Today, this confidence level is not explicitly specified in Grid Codes. Hence, this level is chosen arbitrarily, but the 95% value is commonly found in chance-constrained problems that are met in the literature.

It is interesting to note that the selection of weights $\omega_1, \omega_2, \omega_3, \omega_4$ is important as the objectives are clearly antagonistic. For example, decreasing the control efforts contradicts with maximizing the worst PQ confidence level κ_i , which calls for a study of a Pareto barrier.

4.2. Results

It is impossible to prove formally the convexity of a problem using numerical simulations, but, similar to the Monte Carlo method, running a high number of simulations with random starting points makes it possible to infer, in practice, whether the problem is convex. This subsection displays the results of a numerical study of the convexity and the convergence speed of the proposed optimization problem using this industrial example.

The optimization problem (24) is more complex than the theoretical max-p programming problem, but can nevertheless be solved by the same approach as in section 3.2, and is therefore convex. Using a standard PC equipped with Intel Core processor i7-4810MQ and 16 Go RAM, and Matlab2016b interior

point routines, procedure 2 solves the optimization problem applied on a feeder within 30 seconds. The optimization applied on the full industrial grid in which the feeder belongs to, with 3446 nodes, 27 feeders and 21 DGs is achieved in 8-12 minutes. This computational time is consistent with the industrial time range and an hourly update of DGs control parameters.

The objective function tolerance is 10^{-2} , which means that the search algorithm stops when the decrement of the objective function between two consecutive iterations is below this value. The numerical study of the convexity was achieved by running 1000 optimizations using random starting points. As a result, all optimizations converge in less than 10 iterations and all the solutions are grouped around the objective function value 0.9005. Fig.6 presents the histogram of the objective function values and displays the domain $\mathcal{B} = [0.9005 - 0.5 \times 10^{-2}; 0.9005 + 0.5 \times 10^{-2}]$ which is centered on the most probable solution 0.9005 and which width equals the tolerance. As the number of iterations is small, this study shows that the convex optimization algorithm can reach the global minimum quite accurately and fast.

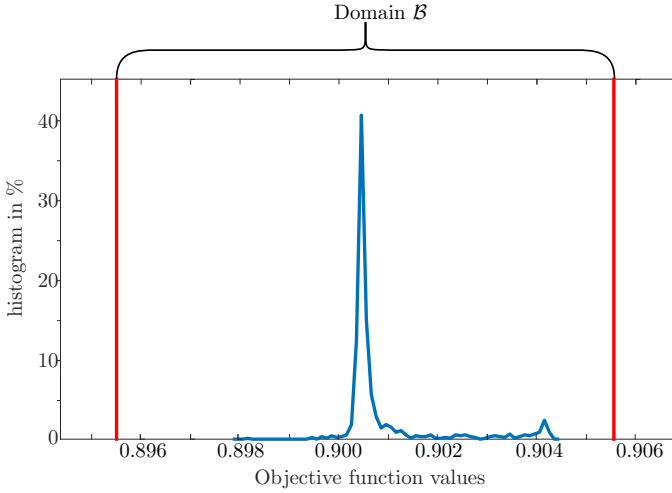


Figure 6. Histogram of objective function value

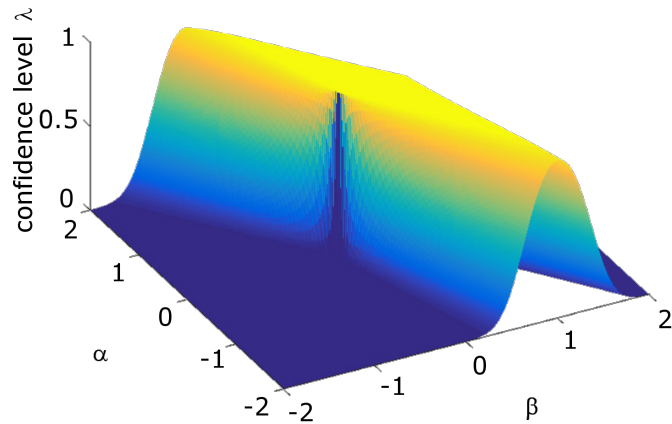


Figure 7. Confidence level for PQ diagrams

Fig. 7 shows an example of a confidence level that the DG

power comply with the PQ domain specifications, as a function of parameters $\mathbf{x}^\alpha, \mathbf{x}^\beta$. One can see that a region of the parameters set is able to reach at least 95% confidence levels. This figure shows the feasible region created by this constraint is convex. This result can be extended to the feasible region created by all the constraints (25).

The results from the optimization problem that minimizes (24) s.t. (25) can be compared with classical control tuning methods. The first method assumed no reactive power control that is $\tilde{\mathbf{Q}}^p = 0$. A second method uses a maximum $\tan(\phi)$ for all DGs that is, for all generators, $\tilde{\mathbf{Q}}^p = -0.35\tilde{\mathbf{P}}^p$, and $\alpha_{ii} = \mathcal{Q}_i^0 = 0$. This control law yields the minimum voltages variances. The third control law assumes that the reactive powers of all DGs achieve a maximal effort $\tilde{\mathbf{Q}}^p = -0.35pu$, and yield the minimum voltages expectations.

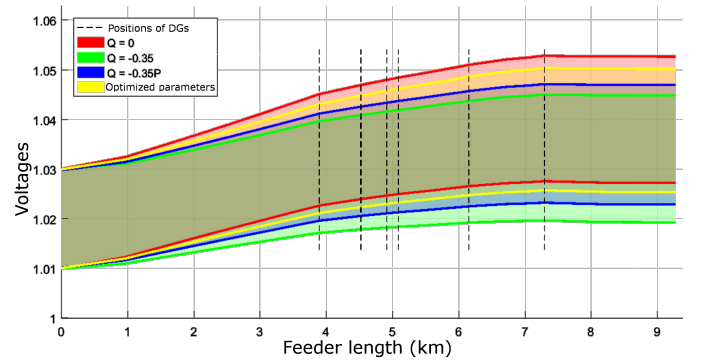


Figure 8. Voltage 95 % confidence levels along the feeder

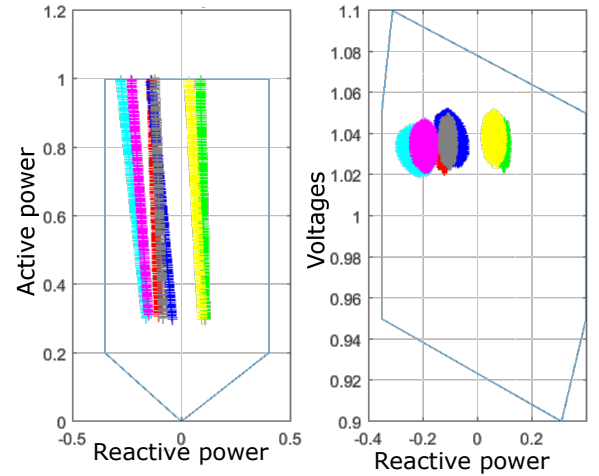


Figure 9. PQ diagrams and PQ 95 % confidence levels with optimal parameters

Fig. 8 presents the confidence levels that the voltages along the feeder lie in the contractual interval for all three methods and the optimization problem. Fig. 9 shows 95 % confidence intervals of PQ and QV diagrams. Finally, Table 2 show the values of means and standard deviations of voltages at the middle and end of the feeder and Table 3 displays the subcriteria for each classical and optimized controller parameter, that is mini-

mal confidence levels, control efforts and voltage variances.

Method	Mid-feeder voltages		End-Feeder voltages	
	Mean (pu)	Std (pu)	Mean (pu)	Std (pu)
$Q = 0, V_0 = 1.02pu$	1.0364	0.00588	1.0399	0.00644
$Q = -0.35, V_0 = 1.02pu$	1.0300	0.00585	1.0323	0.00640
$Q = -0.35P, V_0 = 1.02pu$	1.0323	0.00565	1.0349	0.00608
Optimized parameters	1.0345	0.00570	1.0378	0.00621

Table 2. Optimized non-optimal voltage means and standard deviations

Method	Node Voltage end of feeder		$\min_i(\eta_i)$	$\sum_i E[\frac{\bar{P}_i}{P_i}]$	Global Objective function value
	Mean (pu)	Std (pu)			
$Q = 0, V_0 = 1.02pu$	1.0399	0.00644	93.1 %	0	0.9199
$Q = -0.35, V_0 = 1.02pu$	1.0323	0.00640	99.7 %	0.35	0.8106
$Q = -0.35P, V_0 = 1.02pu$	1.0349	0.00608	99.2 %	0.2243	0.8688
Optimized linear law	1.0382	0.00626	97.5 %	0.1270	0.9005

Table 3. Values of subcriteria

4.3. Discussion : Weighting factors selection

Fig. 8 shows that, without any reactive power control, the node voltages are unable to meet their specifications. Typically, at the end of the feeder, the overvoltage risk is 6.9 % whereas it should not exceed 5 %. Traditional control laws are not flexible and generate very secure working conditions (more than 99 % minimum confidence level at the lowest voltage node) at the expense of high DG control efforts. Typically, in the case study, control efforts from minimum $\tan(\phi)$ and maximum reactive power are respectively twice and three times that of the proposed optimized settings. It is interesting to note that using the minimum $\tan(\phi)$ law allows to minimize the voltage variance as expected. Note that indeed, the optimal strategy maximizes the objective function, with the exception of the zero reactive power control strategy, which main objective consists to prevent constraints violations.

The optimal strategy is indeed able to realize a compromise between subcriteria which depends on the weighting factors $\omega_1, \omega_2, \omega_3, \omega_4$, showing good results on variances and confidence levels while requiring a reasonable control effort. The weighting factors ω_i can be tuned by different methods, for example by drawing a Pareto barrier. An illustration given in Fig. 10 is done between the DG efforts criteria and the maximum risk that voltages lies out the domain $[0.95; 1.05]$ pu ($\max_{i=0 \dots n} 1 - \eta_i = 1 - \min_{i=0 \dots n} \eta_i$). This figure allows to choose the values of ω_1 and ω_4 . The same study has been done for all weighting factors, in our case $\omega = (0.995; 0.995; 1; 0.0005)$.

5. Conclusion

The insertion of intermittent renewable sources into distribution grids generates bidirectional power flows and violations of

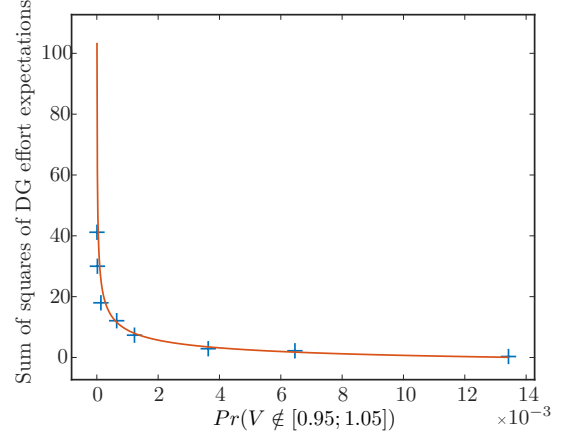


Figure 10. Pareto barrier

the voltage specifications when only local controllers are operating. The design of an accurate linear stochastic model of the grid allows to represent the stochastic nature of the powers and voltages. Using this model, it is possible to find the solution of a centralized optimization problem which yields the appropriate parameters of the affine laws of the DG reactive power controller. The optimal criterion realizes a trade-off between the risk of voltage overshoot or that the powers leave a prescribed PQ domain and the control efforts required from the DGs. It is shown, using a real industrial feeder, that this method yields a good confidence level, and thus a good operating performance, at a reasonable energetic cost. An algorithmic procedure is given to solve this complex problem which is relaxed and transformed in a sequence of convex subproblems. Contrary to classical OPF formulations, the present convex formulation allows to guarantee a simple and fast resolution. The methodology is realistic as it allows to update the control hourly, the optimal supervisor requiring only a simplified model of the system and the characteristics of the stochastic laws of the productions and consumptions (assuming to be Gaussian). Moreover, the industrial control structure is kept and, after the occurrence of an update, the controllers continue to operate in a decentralized way, which does not require inter-communication. The algorithm is able to complete an update in less than 10 minutes, which is not prohibitive as for Monte Carlo methods. Future work will focus on more complicated industrial control laws, for example which consider a dead-band.

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