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Many different fatigue criteria are available in the literature and can be used by the designer who deals with multiaxial fatigue. The choice of the most appropriate one is an arduous task but sometimes it can be proved that criteria built according to different approaches can lead to very similar predictions. Most of the fatigue criteria can be divided into two groups. The criteria of the first one are stress or strain based and some of them use a critical plane concept. The second one, composed of energy based criteria, can be subdivided into (i) critical plane energy based approaches and (ii) global energy based methods. A comprehensive survey of energy criteria was published in 1999 by Macha and Sonsino [1].

Global energy based approaches

A usual approach for ductile materials in high cycle fatigue (HCF) under proportional multiaxial loadings [1] is the distorsion strain energy hypothesis, known as Von Mises hypothesis. It considers that fatigue damage is the consequence of the deviatoric (distorsion) strain energy density $\Phi_{d,a} = s_{ij,a}e_{ij,a}/2$, where $s_{ij,a}$ and $e_{ij,a}$ are respectively the alternated parts of the deviatoric stress and strain tensors.

In 1974, Ellyin [2] proposed a fatigue criterion based on the cyclic shear strain energy $W_s$ defined by: $W_s = \int_{cycle} s_{ij}\,d\varepsilon_{ij}$. This parameter is a sum of elastic and plastic shear strain energies. Several evolutions of this criterion were proposed. Lefebvre et al. [3] used the effective strain energy while Ellyin and Golos [4] considered the sum of elastic strain energy under tension and plastic energy of effective strain. Leis [5] assumed that the damaging energy parameter under proportional multiaxial fatigue with creep is the total internal strain energy $W_t = \int_{cycle} \Sigma_{ij}dE_{ij}$. This parameter includes also the plastic and the elastic strain energy density. Garud [6] assumed that plastic strain energy $W_p$ is a relevant damage parameter to predict fatigue crack initiation. $W_p = \int_{cycle} \Sigma_{ij}dE_{ij}$. This energy is the sum of the normal and shear plastic strain. In 1998, Palin-Luc et al. [7] proposed, for HCF under fully reversed loadings, a volumetric energy based
multiaxial fatigue criterion considering that the damage parameter is the mean value over a loading cycle of the potential elastic strain energy density: \( W_* = \frac{1}{2} \int \Sigma_{ijkl} E_{ijkl} \, dt \). From this proposal, Banville and Palin-Luc proposed in 2001 [7] - [9] an evolution of this criterion for any loadings. The damaging parameter is then the elastic strain work density (after elastic shakedown which is supposed to occur in middle cycle fatigue and high cycle fatigue [10]), \( W_* \), given to an elementary volume by each component of the stress strain tensors: \( W_* = \sum_n \frac{1}{2} \left( \Sigma_{ij}(t) \cdot E_{ij}(t) \right) \, dt \).

**Critical plane energy based approaches**

Smith, Watson and Topper (SWT) [11] proposed to take into account the mean stress effect on the tensile fatigue strength by the product of the normal strain amplitude and the maximum normal stress. Recently, Socie [12] observed that short fatigue cracks grow on the plane perpendicular to the maximum normal stress, \( \sigma_n \). The critical plane is the material plane experiencing the maximum value of the previous sum. But, according to the authors, this parameter has not been clearly correlated with experimental results. In 1993, Liu proposed to predict the multiaxial fatigue strength of materials under proportional and non-proportional loadings by using the virtual strain energy (VSE) \( \Delta W_{VSE} = \Delta W_{VSE,n} + \Delta W_{VSE,y} \), \( \Delta \Sigma \Delta \varepsilon + \Delta \tau \Delta \gamma / 2 \) computed on the critical fracture plane. This critical plane is depending on the cracking mode of the material (mode I or II) and on the crack orientation in mode II (type A or B) [14].

Glinka et al. [15] used the normal and shear parts of energy in the critical plane: \( W^* = \left( \Delta \sigma \Delta \varepsilon / 2 \right) + \left( \Delta \tau / \Delta \gamma / 2 \right) \). This plane experiences the maximum shear strain.

The previous review is not exhaustive, a lot of energy based parameters being available to estimate fatigue life under variable amplitude loadings. Indeed, a cycle counting algorithm is often applied to a counting variable and it is easier to extract cycles by handling a scalar than by handling a stress vector. An other reason is that the product of stress and strain related to a material plane seems appropriate to take into account the multiaxiality of stresses and strains and to predict the fatigue crack initiation or fracture plane orientation. Furthermore, by considering both stresses and strains, the plastic flow which occurs in LCF is taken into account, a combination of stress and strain is - a priori - interesting for an approach devoted to both LCF and HCF.

**Global Strain Energy and Critical Plane Energy Parameters**

Let us define now several energy parameters to be used in the next parts. A point \( O \) of a loaded component and a frame \( \{O;\Sigma,\gamma,\tau\} \) attached to the component systems linked to the material plane \( \Delta \) (Figure 1).

![Specimen surface](image)

\( \sum_n \frac{1}{2} \left( \Sigma_{ij}(t) \cdot E_{ij}(t) \right) \, dt \) is composed of a normal stress vector \( \sigma(t) \), and a shear stress vector \( \tau(t) \):

\[ \Sigma_n(t) = \sigma(t) + \frac{1}{2} \frac{\Delta \varepsilon}{\Delta \varepsilon} \]

where \( \sigma(t) = (\Sigma_n(t) \cdot n)n \)

\[ \tau(t) = (\Sigma_n(t) - \sigma(t)) \]

Now, if the strain state at the same point \( O \) is defined by the tensor \( \epsilon(t) \), the strain vector \( \epsilon(t) \) on the material plane \( \Delta \) can be defined by:

\[ \epsilon(t) = \frac{1}{2} \epsilon(t) \cdot n \]

This vector is composed of a longitudinal strain vector \( \epsilon(t) \) and a shear part \( \gamma(t) \) as well:

\[ \epsilon(t) = \epsilon(t) \cdot n + \gamma(t) \]

From the aforementioned quantities (eq. 4) and (eq. 7), at any time the strain energy density parameter \( \psi_{\epsilon}(t) \) related to the material plane \( \Delta \) can be defined by:

\[ \psi_{\epsilon}(t) = \Sigma_n(t) \cdot \epsilon(t) \quad \Rightarrow \quad \psi_{\epsilon}(t) = \sigma(t) \cdot \epsilon(t) + \frac{1}{2} \frac{\Delta \varepsilon}{\Delta \varepsilon} \]

If a normal part \( \psi_{\epsilon,n} \) and a shear part \( \psi_{\epsilon,\gamma} \) of the strain energy density parameter \( \psi_{\epsilon} \) for the material plane \( \Delta \) are introduced, it comes:

\[ \psi_{\epsilon,n}(t) = \psi_{\epsilon,n}(t) + \psi_{\epsilon,\gamma}(t) \]

where \( \psi_{\epsilon,n}(t) = \sigma(t) \cdot \epsilon(t) \) and \( \psi_{\epsilon,\gamma}(t) = \frac{1}{2} \frac{\Delta \varepsilon}{\Delta \varepsilon} \cdot \gamma(t) \)

For an homogeneous material with an isotropic elastic behaviour at the macroscopic scale, let us now try to establish the relationships between these energy quantities related to a material plane and the potential elastic strain energy...
density $\Phi(t) = \Sigma(t)E(t)/2$ which can be expressed as the sum of two parts: the distortion $\Phi_d(t)$ and spherical $\Phi_s(t)$ parts defined as:

$$\Phi_d(t) = \frac{1}{E} I_2(t)$$

and

$$\Phi_s(t) = \frac{3}{2} \left( \frac{1-2v}{E} \right) \Sigma^s(t)$$

(12)

where $I_2$ is the second invariant of the deviatoric stress tensor, $\Sigma$ is the hydrostatic stress, $E$ is the Young modulus, and $v$ is the Poisson ratio of the material.

Eqs. 16 and 17 prove that a direct link exists at any time between the different energy parameters on a material plane and the corresponding “global” potential strain energy densities $\Phi_d$ and $\Phi_s$ usually defined. However, while the volumetric average of the shear strain energy parameter related to a plane $\hat{\psi}_{XY}$ is simply proportional to the distortion strain energy density $\Phi_d$, the volumetric average of the normal strain energy density $\hat{\psi}_{XX}$ depends both on the distortion and spherical parts. This tends to indicate that the energy related to a plane must be handled very carefully especially when dealing with its normal part.

After some long calculations, it can be shown that $\hat{\psi}_{XY}$ (respectively $\hat{\psi}_{XX}$) acting on the material plane follows the remark its can be demonstrated the link existing between stress based critical plane fatigue criteria and energy based ones. Let us assume that the stress tensor is periodic with a mean value $\Sigma_{lin}$ and an alternating part $\Sigma_{a}$; its time evolution may be sinusoidal, triangular or square (eq. 18), where “triang” and “sqr” are respectively the fully reversed triangular and square periodic functions versus time.

$$\Sigma(t) = \Sigma_{lin} + \Sigma_{a}$$

(18)

From the previous equations, the alternating and mean value of the normal and shear stress or strain vectors can be defined.

$$\hat{\psi}_{XX} = \hat{\psi}_{XX}^{a} + \hat{\psi}_{XX}^{m}$$

(19)

$$\hat{\psi}_{XY} = \hat{\psi}_{XY}^{a} + \hat{\psi}_{XY}^{m}$$

(20)

After some long calculations, it can be shown that $\hat{\psi}_{XX}$ and $\hat{\psi}_{XY}$ are the sum of the contributions of $\hat{\psi}_{XX}^{a}$ and $\hat{\psi}_{XY}^{a}$ acting on the material plane.

Critical Plane Energy Approach

As demonstrated in 1993 by Papadopoulos [16], for any synchronous multiaxial sinusoidal loading: $\Sigma(t) = \Sigma_{lin} + \Sigma_{a}$ sin $(\omega t + \phi_0)$, the shear stress vector $\hat{\psi}_{XY}$ acting on the material plane $\Delta$ oriented by $\psi_0$ can be written as a combination of $\psi_{XX}$ and $\psi_{XY}$ (Figure 2) (eq. 23) where $A$, $B$, $C$, and $D$ are function of the phase differences $\phi_0$ and of the stress amplitudes $\Sigma_{p,a}$.

$$\Sigma(t) = \Sigma_{lin} + \Sigma_{a}$$

(23)
the macroscopic resolved shear stress amplitude \( T_\alpha \) acting on all the possible sliding directions on one material plane \( \Delta \) (Figure 2).

\[
T_\alpha(\theta, \varphi) = \sqrt{\frac{2}{\pi^2} \int_{\varphi=0}^{\pi} T_\alpha^2(\theta, \varphi, \chi) d\chi}
\]

(25)

Indeed, Papadopoulos showed that \( T_\alpha \) is equal to:

\[
T_\alpha(\theta, \varphi) = \sqrt{\pi \left(A^2 + B^2 + C^2 + D^2\right)}
\]

(26)

Thus, for any synchronous multiaxial sinusoidal loading, eq. 27 relates \( T \) and the alternating shear strain energy density parameter on the material plane \( \Delta \):

\[
T_\alpha(\theta, \varphi) = \sqrt{\frac{2 \pi E}{1 + \nu} \int \frac{1}{\pi} \left| \tau_{\alpha}(t) \right| dt}
\]

(27)

For “ductile” metals (i.e. if \( 0.5 \leq \varepsilon_{\text{st}}/\varepsilon_{\text{stmax}} \leq 0.6 \), Papadopoulos proposed an endurance criterion using \( T_\alpha \) and the maximum hydrostatic stress on a loading period. Since the effect of this last parameter is very important in fatigue crack initiation, the criterion is a linear combination (eq. 28) where \( a \) and \( b \) are two material parameters identified with two experimental fatigue limits.

\[
\max_{\theta, \varphi} \left[ T_\alpha(\theta, \varphi) \right] + a \Sigma_{\alpha,\text{max}} \leq b
\]

(28)

From eq. 16 it is now clear that the “Papadopoulos \( T_\alpha \) criterion” (eq. 28) can be written as follows (eq. 29) with energy parameters, where \( I_1(t) = \Sigma(t) \):

\[
\max_{\theta, \varphi} \left[ T_\alpha(\theta, \varphi) \right] + a \Sigma_{\alpha,\text{max}} \leq b \iff
\max_{\theta, \varphi} \left[ \frac{2 \pi E}{1 + \nu} \int \frac{1}{\pi} \left| \tau_{\alpha}(t) \right| dt \right] + a \max_{\theta, \varphi} \left[ \frac{1}{I_1(t)} \left( \frac{2 \pi E}{1 + \nu} \right) \frac{1}{I_1(t)} \right] \leq b
\]

(29)

For “hard” metals (i.e. if \( 0.6 \leq \varepsilon_{\text{st}}/\varepsilon_{\text{stmax}} \leq 0.8 \)), Papadopoulos proposed another criterion (eq. 30).

\[
M_\alpha + c \Sigma_{\alpha,\text{max}} \leq d
\]

(30)

Where the \( M_\alpha \) quantity is nothing else but the volumetric mean value of \( T_\alpha \) over all the material planes of an elementary volume (eq. 31). This “global” quantity was defined by Papadopoulos because fatigue crack initiation is assumed to be strongly dependent on the shear stress amplitude, not only over all the possible sliding directions on one material plane \( \Delta \) (or a crystal), but also over all the planes of a grain.

\[
M_\alpha = \sqrt{\frac{5}{8 \pi^2} \int \frac{2}{\pi^2} \int T_\alpha^2(\theta, \varphi, \chi) \sin \theta d\theta d\varphi}
\]

(31)

In 2001, Morel et al. [18] proved that for any synchronous sinusoidal multiaxial loadings the second Papadopoulos criterion can be easily written with energy quantities (eq. 32).

\[
M_\alpha + c \Sigma_{\alpha,\text{max}} \leq d \iff \pi \frac{2E}{1 + \nu} W_{d,\alpha} + c \max_{\varphi} \left[ \frac{1}{I_1(t)} \left( \frac{2 \pi E}{1 + \nu} \right) \Phi(t) \right] \leq d
\]

(32)

Where \( W_{d,\alpha} \) is the deviatoric part of the mean value over a loading cycle of the potential strain energy density due to alternated stresses (eq. 33) as used in the volumetric energy based high cycle multiaxial fatigue criterion proposed by Palin-Luc and Lasserre [7].

\[
W_{d,\alpha} = \frac{1}{T^2} \int \Phi_{d,\alpha}(t) dt = \left( \frac{1 + \nu}{E} \right) \frac{1}{T^2} \int I_1(t) dt
\]

(33)

This means that the material planes experiencing the maximum value of \( T_\alpha \) and \( V W_{d,\alpha} \) are the same. As illustrated in Figure 3, in many loading cases these planes are not unique; for in-
Since the critical planes according to $T_0$ or $\sqrt{W_{g,n}}$, are not always unique, it is interesting to consider all the planes of an elementary volume of material. In this case, the shear strain work $W_g$, and the normal strain work $W_n$ given to all the planes within an elementary volume are respectively defined by equations (eq. 37).

$$W_g = \int_{\frac{\pi}{3}}^{\pi} \left( \frac{2}{3} \Phi(t) \right) dt$$

$$W_n = \int_{\frac{\pi}{3}}^{\pi} \left( \frac{2}{5} \Phi(t) \right) dt$$

From eq. 16 it is easy to prove that:

$$W_g = \int_{\frac{2}{3}}^{\frac{2}{5}} \Phi(t) dt$$

$$W_n = \int_{\frac{2}{3}}^{\frac{2}{5}} \Phi(t) dt$$

Eq. 38 means that $W_g$ is also the sum over the load cycle of the positive variations of the distortion part of the potential strain energy density $\Phi(t)$, and $W_n$ is a combination of the positive variations of both the spherical and the deviatoric parts of $\Phi(t)$.

### Discussion

Findley’s experiments. In a well known paper, Findley et al. [19] describe a fatigue test on a rotating disk loaded in diametral compression by rollers. Fatigue cracks initiated at the center of this disk. In this area there was a biaxial stress state and principal direction of stresses were rotating in a frame linked with the specimen. Because, at this location, the potential strain energy density $\Sigma(t) E(t)/2$ was constant over one rotation of the disk, the authors concluded that the energy can not be used to predict fatigue crack initiation. For instance, with a static tension test it is possible to submit a specimen to the same value of the potential strain energy density as during the Findley et al. fatigue test, without crack initiation (the static load required to initiate a crack is higher). Thus for many years, this paper was used by several authors to claim that energy based approaches are not appropriate to predict the fatigue strength of materials and structures. But this was a premature conclusion.

Of course, the potential strain energy density is constant in the previous experiment and the energy parameter proposed by Leis [5] is not relevant to fatigue. However, Park and Nelson [20] demonstrated that the distortion work is not null nor constant for the Findley et al. test. They refute the negative conclusion of Findley et al. concerning energy based approaches for fatigue. On a similar case, Banvillet et al. [10] showed that the strain work density given to the material per loading period is adapted for fatigue prediction. If a critical plane energy based parameter is now considered, the following results are obtained. At the center of the disk of the Findley’s tests, the stress state can be written as [20]:

$$\sigma_{11}(t) = 2.9 \sigma \sin(\pi t) - 1.1 \sigma,$$

$$\sigma_{11}(t) = 2.9 \sigma \sin(\pi t - \pi) - 1.1 \sigma,$$

$$\sigma_{11}(t) = 2.9 \sigma \sin(\pi t - 2\pi)$$

where $\sigma$ is a constant. For such a stress state one can demonstrate that the spherical and deviatoric parts of the potential strain energy density are not time dependent (eq. 39) as shown by Findley et al. for the total potential strain energy density $\Phi(t)$ [3, 7].

$$\Phi_{11}(t) = 2 \left( \frac{1 - 2\nu}{3E} \right) \left[ (1.1\sigma)^2 + (1.1\sigma)^2 \right]$$

$$\Phi_{11}(t) = 2 \left( \frac{1 + \nu}{3E} \right) \left[ 3(2.9\sigma)^2 + (1.1\sigma)^2 \right]$$

In the same way, the normal and shear parts ($\psi_{\pi,\pi}$ and $\psi_{\pi,\gamma}$) of the elastic strain energy density parameter $\psi$ related to a material plane $\Delta$ orientated by $(\theta, \phi)$ are fluctuating with time (eqs. 40 and 41) during the experiment of Findley. $\psi_{\pi,\pi}$ and $\psi_{\pi,\gamma}$ are cyclic during one rotation of the disk (Figure 4, on a particular plane). Thus, these quantities can be used as damage parameters in fatigue. Banvillet et al. obtained the same conclusion with the strain work density due to each stress strain components [10].

$$\psi_{\pi,\pi}(t) = \frac{-\sigma^2 \sin^2 \theta}{2E} \left[ -1.1(1 + 3\nu + (1 + \nu)\sin(2\theta)) + 5.8 (1 + \nu) \sin^2 \theta \sin(2\pi - \omega t) \right]$$

$$\psi_{\pi,\gamma}(t) = \left( \frac{1 + \nu}{4E} \right) \left[ 27.65 + 0.83 \cos(2\theta) \right]$$

$$\psi_{\pi,\gamma}(t) = (1 + \nu) t \sin^2 \theta + 16.82 \cos(4\theta - 2\omega \pi \sin^2 \theta + 6.38 \sin^2 (2\theta) \sin (2\pi - \omega t) \right]$$

However, it can be proved for this test that eqs. 38 and eq. 39 lead to $W_g = W_n = 0$. This means that $W_g$ and $W_n$ are not able to reflect the fatigue damage accumulation for that kind of test.

Application to combined tension and torsion loading. Even if $W_{\gamma,\pi,\gamma}$ and $T_\pi$ are not simply linked under non-proportional loadings, the material planes experiencing the maximum values of these two quantities can be the same (Figure 5). For a combined tension and torsion sinusoidal fatigue test with a phase shift of $90^\circ$, this Figure shows that the critical planes corresponding to $W_{\gamma,\pi,\gamma}$ and $T_\pi$ are the same if $\Sigma_{\pi,\pi}/\Sigma_{\pi,\gamma} = 2$ (Figure 5a), but they are different if $\Sigma_{\pi,\pi}/\Sigma_{\pi,\gamma} = 1/2$ (Figure 5b). Nevertheless, the difference between the maxima of $T_\pi$ in Figure 5b) and its values for the $W_{\gamma,\pi,\gamma}$ critical planes is not important: a lot of planes have a high $T_\pi$ value (between 100 and 120 MPa), many of them are in coincidence with the $W_{\gamma,\pi,\gamma}$ critical planes. Experiments have to be done to try to observe the fatigue crack initiation plane (not the fracture plane) in such test conditions.

Energy based interpretation of fatigue crack initiation at the mesoscale level. In his mesoscopic description of fatigue crack initiation in a polycrystalline material, Papadopoulos [17] assumed that the plasticity criterion of each grain is the Schmid’s law and that each crystal follows a combined isotropic and kinematical hardening (modulus $g$) behaviour. Under the assumption that, at the endurance limit, the plastically deformed crystal reaches an elastic shake-down state, this author found, for “ductile metals” (i.e. if
0.5 ≤ τ/σ_{\text{eq},\text{min}} ≤ 0.6), an upper bound estimation, $T_{\text{eq}}/\sigma$, of the plastic strain accumulated by the plastically less resistant grains of the elementary volume (grains with their easier glide planes parallel to the critical plane $\Delta$). Eq. 27 proves that, for synchronous multiaxial sinusoidal loadings, this threshold value of the mesoscopic plastic strain accumulated is also proportional to the square root of the mean value, over a loading period, of the alternating part of the shear strain energy parameter on the critical plane $\psi_{\Delta}$. This equation shows also, that the integration carried out over a loading cycle of an energy based parameter can lead to results similar to an integration of the resolved shear stress amplitude over each possible slip direction of the critical plane. The link between integration in time of an energy based quantity and integration in space on an elementary plane has already been noticed by Morel et al. [18]. This is illustrated by eq. 31 and eq. 32 which prove that the integration, over a loading cycle, of the distortion potential strain energy density due to the alternated stresses gives a result proportional to the integration of the resolved shear stress amplitude over each possible slip direction of every material planes in a point.

For “hard metals” (i.e. if $0.6 ≤ \tau/\sigma_{\text{eq},\text{min}} ≤ 0.8$) Papadopoulos proved that $M_{\text{eq}}/\sigma$ is an upper bound limitation of the mesoscopic plastic strain accumulated by all the gliding crystals within an elementary volume. Eq. 32 shows that for any synchronous multiaxial loadings, $\sqrt{W_{\Delta}}$ can be also considered as an upper bound estimation of the previous quantity since it is proportional to $M_{\text{eq}}/\sigma$. Banville shows [8] that the strain work density given to an elementary volume of material $W_{\sigma} = \Sigma_{ij}(\Sigma_{ij}(t)E_{i}\cdot E_{j})\, \text{d}t$, is proportional to the mean value over a loading period of the potential elastic strain energy density $W_{\sigma} = W_{\Delta}$ for any loadings, except for out-of-phase biaxial tension (for phase shift different from 0 and $\pi$) [8]. It is shown in [18] that $W_{\sigma}$ is the sum of a spherical and a deviatoric parts: $W_{\sigma} = W_{\sigma,s} + W_{\sigma,d}$. For $W_{\sigma}$, the same type of relation can be demonstrated ($W_{\sigma} = W_{\Delta} + W_{\sigma,d}$), except for out-of-phase biaxial tension. This leads to: $W_{\Delta} = 4\, W_{\sigma,d}$, except for out-of-phase biaxial tension.

As a consequence, $\sqrt{W_{\Delta}}$ can be also considered as an upper bound limit of the mesoscopic plastic strain accumulated by all the gliding crystals within an elementary volume under cyclic loading, except in out-of-phase biaxial tension. In future works, these laboratory test conditions have to be studied carefully to propose a new definition of the strain work energy density given to the material under out-of-phase biaxial tension.

### Conclusion and prospects

This paper shows that, in a lot of multiaxial fatigue test conditions, several critical plane stress quantities can be related to critical plane energy based parameters or global energy based ones. Moreover, several new energy parameters related to a material plane have been introduced. By a proper integration over all the material planes, these quantities can be linked to the well-known deviatoric and spherical strain energy densities. The main important result of this work is that the mesoscopic critical plane endurance criteria proposed by Papadopoulos can be readily interpreted in terms of energy quantity. Of course, many experimental investigations under multiaxial loading have still to be done especially under out-of-phase biaxial tension. Such tests would allow to discriminate stress (or strain) and energy (or work) approaches for instance through the influence of the phase shift on the fatigue life. But this work is very promising especially if the low cycle fatigue regime is planned to be investigated. Indeed, in this fatigue regime, the use of an energy approach seems the more appropriate because both the stress and the strain quantities can play an important role in the damage mechanisms.

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**Nomenclature**

- $O_{1,2,3}$: orthogonal cartesian coordinate system linked to the component $\Delta$
- $\varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3}$: strain vector acting on the material plane $\Delta$
- $\Phi_{1}, \Phi_{2}, \Phi_{3}$: strain vector acting on the material plane $\Delta$
- $\rho, \theta, \phi$: unit normal vector orientating the material plane $\Delta$
- $\theta, \pi$: spherical co-ordinates of unit normal vector $\rho$
- $\Omega$: Poisson ratio
- $\nu$: normal stress vector acting on the material plane $\Delta$
- $\gamma$: shear stress vector acting on the material plane $\Delta$
- $\xi$: normal strain vector acting on the material plane $\Delta$
- $\Phi$: volumetric density of the potential elastic strain energy
- $\Phi_{0}, \Phi_{1}, \Phi_{2}$: distortion part of $\Phi$
- $\psi_{\Delta}$: strain energy density parameter defined on the material plane $\Delta$ orientated by $\rho$
- $W_{\sigma}$: normal part of $W_{\sigma}$
- $W_{\sigma,s}$: shear part of $W_{\sigma}$
- $\Sigma_{ij}$: second invariant of the stress deviatoric tensor
- $I_{1}$: first invariant of the shear stress tensor, $I_{1} = \Sigma_{ij}\varepsilon_{ij}$
- $S_{\sigma_{\text{eq},\text{min}}}$: fully reversed endurance limit in tension
- $S_{\tau_{\text{min}}}$: fully reversed endurance limit in torsion
- $T_{\sigma}$: mean value of the macroscopic resolved shear stress amplitude on the material plane
- $M_{\sigma}$: volumetric mean value of $T_{\sigma}$ over all the material planes
- $a, b$: material parameters of the Papadopoulos $T_{\sigma}$ criterion
- $c, d$: material parameters of the Papadopoulos $M_{\sigma}$ criterion

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Figure 5. $W_{\Delta} \times 10^{6}$ and $T_{\sigma}$ distributions on the different planes oriented by $[\theta, \phi]$, for two combined tension and torsion sinusoidal out-of-phase ($\phi = \pi/2$) fatigue tests
Thus, since 

\[ \phi(t) = \frac{1}{2} \mathbb{E}_{ij}(t) \mathbb{E}_{ij}^{\dagger}(t) \]

\[ \phi(t) = \frac{1}{2} \text{tr}(\mathbb{E}(t) \mathbb{E}^{\dagger}(t)) \]

For an isotropic material with a linear elastic behaviour it is easy to demonstrate that:

\[ \text{APPENDICES} \]

Annex 1: Calculation of \( \psi_{\phi} \) and \( \psi_{\phi_y} \). For any multiaxial loading of an isotropic material with a linear elastic behaviour, the distortion part \( \Phi_d(t) \) and the spherical part \( \Phi_s(t) \) of the potential elastic strain energy density \( \Phi(t) = \mathbb{E}_{ij}(t) \mathbb{E}_{ij}^{\dagger}(t)/2 \) are:

\[ \Phi_d(t) = \frac{1}{2v} \mathbb{I}(t) = \frac{1}{2v} \int \left[ \sum_{\alpha=1}^{6} (\psi_{\phi}(t) + \psi_{\phi_y}(t)) \right] \]

\[ \Phi_s(t) = \frac{1}{2} \left( 1 - 2v \right) \int \left[ \sum_{\alpha=1}^{6} (\nu \psi_{\phi}(t) + \nu \psi_{\phi_y}(t)) \right] \]

On the material plane oriented by \( n \), the shear stress vector \( \tau(t) \) and the shear strain vector \( \gamma(t) \) are respectively:

\[ \mathbb{T}(t) = \mathbb{E}_{ij}(t) - \mathbb{E}_{ji}(t) \]

\[ \mathbb{G}(t) = \mathbb{E}_{ij}(t) - \mathbb{E}_{ji}(t) \]

Their components have the following expressions in the frame \( R = (O; \hat{x}, \hat{y}, \hat{z}) \).

Note: to simplify the following equations, \( t \) is omitted for each \( \sigma_{ij} \) and \( E_{ij} \) term.

\[ \mathbb{G}(t) = \begin{bmatrix} E_x \cos \theta + E_y \cos \phi \sin \theta & E_y \cos \phi \cos \theta & E_z \cos \phi \sin \theta \\ E_x \sin \theta \cos \phi - E_y \cos \theta \sin \phi \cos \theta & E_y \sin \phi \cos \theta + E_z \sin \phi \sin \theta & E_z \sin \phi \sin \theta \\ E_x \sin \phi \cos \theta + E_y \cos \theta \sin \phi \cos \theta & E_y \sin \phi \sin \theta - E_z \cos \phi \sin \theta & E_z \cos \phi \sin \theta \end{bmatrix} \]

The expression of \( \mathbb{G}(t) \) is obtained from the equation of \( \mathbb{E}(t) \) by changing \( \Sigma_y \) in \( \Sigma_y \).

By using the shear stress vector definition: \( \mathbb{T}(t) = \mathbb{E}_{ij}(t) - \mathbb{E}_{ji}(t) \), and the shear strain vector definition: \( \mathbb{G}(t) = \mathbb{E}_{ij}(t) - \mathbb{E}_{ji}(t) \), and the linear elastic behaviour of the material one can compute the following integral:

\[ \int_{\hat{x}=0}^{2\pi} \int_{\hat{y}=0}^{2\pi} \mathbb{G}(t) \mathbb{G}(t)^\top \, d\psi = \frac{4v+1}{4\pi} \left[ \begin{array}{c} \frac{7}{8} S_{xx} + \frac{5}{2} S_{yy} + 4 S_{zz} - \frac{3}{4} S_{xx} S_{yy} + \frac{7}{8} S_{yy} + 4 S_{zz} - S_{xx} S_{yy} + 2 S_{zz} \\ \frac{7}{8} S_{yy} + \frac{5}{2} S_{zz} + 4 S_{xx} - \frac{3}{4} S_{xx} S_{yy} + \frac{7}{8} S_{yy} + 4 S_{zz} - S_{xx} S_{yy} + 2 S_{zz} \\ \frac{7}{8} S_{zz} + \frac{5}{2} S_{xx} + 4 S_{yy} - \frac{3}{4} S_{xx} S_{yy} + \frac{7}{8} S_{yy} + 4 S_{zz} - S_{xx} S_{yy} + 2 S_{zz} \end{array} \right] \cos \theta \]

Then,

\[ \int_{\hat{x}=0}^{2\pi} \int_{\hat{y}=0}^{2\pi} \mathbb{G}(t) \mathbb{G}(t)^\top \, d\psi = \frac{2v+1}{15} \left[ \begin{array}{c} S_{xx} + S_{yy} + S_{zz} + 3 (S_{xx} + S_{yy} + S_{zz}) - (S_{xx} S_{yy} + S_{yy} S_{zz} + S_{xx} S_{yy}) \\ S_{xx} + S_{yy} + S_{zz} + 3 (S_{xx} + S_{yy} + S_{zz}) - (S_{xx} S_{yy} + S_{yy} S_{zz} + S_{xx} S_{yy}) \\ S_{xx} + S_{yy} + S_{zz} + 3 (S_{xx} + S_{yy} + S_{zz}) - (S_{xx} S_{yy} + S_{yy} S_{zz} + S_{xx} S_{yy}) \end{array} \right] \]

So, \( \psi_{\phi_y} = \frac{2}{5} \phi \).

For an isotropic material with a linear elastic behaviour it is easy to demonstrate that:

\[ \mathbb{G}(t) = \frac{1 + v}{E} \sigma - \frac{3v}{E} \Sigma_y \sigma(t) \]

Then, since

\[ 1 = \int_{\hat{x}=0}^{2\pi} \int_{\hat{y}=0}^{2\pi} \sigma(t) \sin \theta \, d\psi = \Sigma_y \]

and

\[ 1 = \int_{\hat{x}=0}^{2\pi} \int_{\hat{y}=0}^{2\pi} \mathbb{G}(t) \sin \theta \, d\psi = \frac{1}{15} \left[ \begin{array}{c} 3 (S_{xx} + S_{yy} + S_{zz}) + 4 (S_{xx} + S_{yy} + S_{zz}) + 2 (S_{xx} S_{yy} + S_{yy} S_{zz} + S_{xx} S_{yy}) \\ 3 (S_{xx} + S_{yy} + S_{zz}) + 4 (S_{xx} + S_{yy} + S_{zz}) + 2 (S_{xx} S_{yy} + S_{yy} S_{zz} + S_{xx} S_{yy}) \end{array} \right] \]

\[ \psi_{\phi_y} = \frac{1}{4\pi} \int_{\hat{x}=0}^{2\pi} \int_{\hat{y}=0}^{2\pi} \mathbb{G}(t) \cdot \mathbb{G}(t) \sin \theta \, d\psi \]

\[ \psi_{\phi_y} = \frac{1}{15} \left[ 3 (S_{xx} + S_{yy} + S_{zz}) + 4 (S_{xx} + S_{yy} + S_{zz}) + 2 (S_{xx} S_{yy} + S_{yy} S_{zz} + S_{xx} S_{yy}) \right] - \frac{3v}{E} \left( S_{xx} + S_{yy} + S_{zz} \right)^2 \]

Thus,

\[ \psi_{\phi_y} = \frac{2}{3} \left( \phi + \frac{2}{5} \phi \right) \]
Annex 2: Calculation of \( \frac{1}{T} \int \psi_{Y,a}(t) \, dt \). As explained in the paper, Papadopoulos demonstrated that in the frame \((O; \mathbf{u}, \mathbf{v}, \mathbf{w})\) for any synchronous multiaxial loading: \( \Sigma(t) = \Sigma_{u,m} + \Sigma_{v,m} \sin(\omega t + \phi) \), the shear stress vector \( \tau(t) \), acting on the material plane \( \Delta \) orientated by \( \mathbf{u} \), can be written as

\[
\tau(t) = T_u(t) \mathbf{u} + T_v(t) \mathbf{v} \quad \text{with} \quad T_u(t) = A \sin \omega t + B \cos \omega t + T_{u,m} \quad \text{and} \quad T_v(t) = C \sin \omega t + D \cos \omega t + T_{v,m}
\]

where \( A, B, C, \) and \( D \) are function of the phase difference \( \phi \) and of the stress amplitudes \( \Sigma_{u,a} \) [16]:

\[
A = \sin(\theta) \left( \frac{\Sigma_{u,a}}{2} \cos(\theta) - \frac{\Sigma_{v,a}}{2} \cos(\theta) \right) + \cos(\theta) \left( \frac{\Sigma_{u,a}}{2} \sin(\theta) \cos(\gamma) + \frac{\Sigma_{v,a}}{2} \sin(\theta) \cos(\gamma) \right)
\]

\[
B = \sin(\theta) \left( \frac{\Sigma_{u,a}}{2} \sin(\theta) - \frac{\Sigma_{v,a}}{2} \sin(\theta) \right) \cos(\gamma) + \cos(\theta) \left( \frac{\Sigma_{u,a}}{2} \sin(\theta) \sin(\gamma) - \frac{\Sigma_{v,a}}{2} \sin(\theta) \sin(\gamma) \right)
\]

\[
C = -\sin(\theta) \left( \frac{\Sigma_{u,a}}{2} \cos(\phi_a) \cos(\gamma) + \frac{\Sigma_{v,a}}{2} \cos(\phi_a) \cos(\gamma) \right) - \cos(\theta) \left( \frac{\Sigma_{u,a}}{2} \sin(\phi_a) \sin(\gamma) - \frac{\Sigma_{v,a}}{2} \sin(\phi_a) \sin(\gamma) \right)
\]

\[
D = \sin(\theta) \left( \frac{\Sigma_{u,a}}{2} \sin(\phi_a) \cos(\gamma) + \frac{\Sigma_{v,a}}{2} \sin(\phi_a) \cos(\gamma) \right) - \cos(\theta) \left( \frac{\Sigma_{u,a}}{2} \sin(\phi_a) \sin(\gamma) - \frac{\Sigma_{v,a}}{2} \sin(\phi_a) \sin(\gamma) \right)
\]

In the frame \((O; \mathbf{u}, \mathbf{v}, \mathbf{w})\), \( T_{u,m} \) and \( T_{v,m} \) are the coordinates of the mean (or static) shear stress vector; \( A \sin \omega t + B \cos \omega t \) and \( C \sin \omega t + D \cos \omega t \) are the coordinates of the dynamic (or alternated) part of the shear stress vector.

From these relations the scalar product \( \tau(t) \cdot \tau(t) \) is: \( \tau(t) \cdot \tau(t) = T_u(t)^2 + T_v(t)^2 \) with \( T_u(t) = A \sin \omega t + B \cos \omega t + T_{u,m} \) and \( T_v(t) = C \sin \omega t + D \cos \omega t + T_{v,m} \).

The integration over a cycle leads to:

\[
\frac{1}{T} \int_0^T \tau(t)^2 \, dt = \frac{A^2}{2} + \frac{B^2}{2} + T_{u,m}^2 \quad \text{and} \quad \frac{1}{T} \int_0^T \tau(t)^2 \, dt = \frac{C^2}{2} + \frac{D^2}{2} + T_{v,m}^2.
\]

Finally,

\[
\frac{1}{T} \int_0^T \tau(t) \cdot \tau(t) \, dt = \frac{1}{T} \int_0^T \tau(t)^2 \, dt + \frac{1}{T} \int_0^T \tau(t)^2 \, dt = \frac{A^2}{2} + \frac{B^2}{2} + \frac{C^2}{2} + \frac{D^2}{2} + T_{u,m}^2 + T_{v,m}^2.
\]

The last two terms \( T_{u,m}^2 + T_{v,m}^2 \) represent to the mean value of the shear stress vector. In other words, they are due to the static part of the loading, while \( \left( \frac{A^2}{2} + \frac{B^2}{2} + \frac{C^2}{2} + \frac{D^2}{2} \right) \) corresponds to the dynamic part of the loading.

Moreover, for an isotropic material with an elastic linear behaviour it comes:

\[
\Sigma(t) = \frac{1}{E} \cdot \tau(t)
\]

So,

\[
\frac{1}{T} \int_0^T \psi_{Y,a}(t) \, dt = \frac{1}{T} \int_0^T \tau(t) \cdot \Sigma(t) \, dt = \frac{1}{T} \int_0^T \left( \frac{A^2 + B^2 + C^2 + D^2}{2} \right) \, dt.
\]

Annex 3: Calculation of the shear work given on a plane. For an homogeneous isotropic linear elastic material the amplitudes of the shear stress and the shear strain vectors are related by: \( \gamma_a(t) = \frac{1}{E} \cdot \tau_a(t) \).

From Annex 2 it is easy to prove that:

\[
\tau_a(t) \gamma_a(t) = \frac{1}{E} \left[ \tau_{u,a}(t) \gamma_{u,a}(t) + \tau_{v,a}(t) \gamma_{v,a}(t) \right] \quad \text{with} \quad T_{u,a}(t) = A \sin \omega t + B \cos \omega t \quad \text{and} \quad T_{v,a}(t) = C \sin \omega t + D \cos \omega t.
\]

Thus, \( \tau_a(t) \gamma_a(t) = \left( \frac{1}{E} \right) \omega \left[ G \sin(2\omega t) + H \cos(2\omega t) \right] \) with \( G = (A^2 - B^2 + C^2 - D^2)/2 \) and \( H = (AB + CD) \).

So

\[
\tau_a(t) \gamma_a(t) = \left( \frac{1 + v}{E} \right) \omega \left[ G \sin(\omega t) + H \cos(\omega t) \right].
\]

From this last equation it comes:

\[
W_{G \gamma_a} = \int_0^T \tau_a(t) \gamma_a(t) \, dt = \frac{4(1 + v)}{E} \sqrt{G^2 + H^2}.
\]
Abstract