Mathematical and numerical analysis in

thermo-gradient-dependent theory of plasticity *

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Abstract

In this paper, we develop new governing equations for thermo-gradient-dependent theory of plasticity. They include the coupled effects of thermal elastic-plastic theory, including balance and constitutive equations. To demonstrate the salient feature of the gradient-dependent model of plasticity, particular attention is addressed to isotropic hardening with second sound effects to eliminate the paradox of infinite speed of thermal signals. The resulting system of partial differential equations formally describes the coupled thermomechanical behavior of the gradient-dependent elasto-plastic system. Then, we develop an appropriate state-space form and, by using the semigroup theory, we prove the well-posedness and the exponential stability of the thermo-gradient-dependent elasto-plastic one-dimensional problem. **Finally, we perform numerical simulations to validate the proposed model and to show its capability.**

1 Introduction

Since classical plasticity usually ignores the effect of the microstructure and its evolution in the course of plastic deformation, consequently, it cannot be used to adequately address the problems related to localization of deformation. On the other hand, discontinuous deformation processes, which cannot be described with classical continuum models, are caused by microstructural phenomena that occur in a localization zone. This gives

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rise to a large variety of generalized continuum models based on well-established continua [1, 2], as well as on more recent plastic strain gradient approaches, to avoid the difficulties in localization simulation of single phase materials. In this context, gradient-dependent models have been recognized by several authors to provide a satisfactory framework for the analytical and numerical analysis of strain localization in single phase solids. In the approach followed in this paper, we include the second-order strain gradient terms in the stress-strain law. The use of a higher-order gradient model results in a well-posed set of partial differential equations.

One of the main objectives of this paper is to derive the thermomechanical theory for gradient-dependent plastic materials. One can find many propositions in the literature for the introduction of thermodynamics into plasticity (see e.g. [3, 4, 5, 6, 7]). **A thermodynamics based higher-order gradient theory for size dependent plasticity was proposed in [3], but the thermal effects were not addressed. More recently, the thermal effects have been introduced in relevant strain gradient plasticity models within the thermodynamically consistent framework ([4, 5, 6, 7]). In [5, 6], Voyiadjis *et al.* developed a coupled thermo-mechanical gradient enhanced continuum plasticity theory. In these works, the higher-order thermo-mechanical gradient plasticity theory is developed within the thermodynamically consistent framework based on the concept of thermal activation energy and dislocations interaction mechanisms, and the decomposition of the thermodynamic microforces into energetic and dissipative counterparts. The effect of the passivation on the higher order gradient plasticity models for the non-proportional loading condition is then examined in terms of some specific phenomenon, which is called as Stress jump. In this paper, a coupled thermo-mechanical gradient enhanced continuum plasticity theory is built. The proposed model is developed in the same spirit as the models presented in ([5, 6, 7]), and its thermodynamic consistency is checked.**

The usual theory of heat conduction based on Fourier's law (see Eq. (3.20)) allows thermal waves to propagate with infinite speed, which is not well accepted from a physical point of view. This is referred to as the paradox of heat conduction. In contrast to conventional Fourier's law, generalized laws came into existence during the last decades to overcome this paradox, and, at the same time, to well describe phenomena arising at very low temperatures, such as the "second sound". These models are based on hyperbolic-type equations for temperature and are closely connected with the theories of second sound, which view heat propagation as a wave-like phenomenon. In an idealized solid, for example, the thermal energy can be transported by quantized electronic excitations, which are called free electrons, and by the quanta of lattice vibrations, which are called phonons. These quanta undergo collisions of a dissipative nature, causing a thermal resistance in the medium. A relaxation time is associated with the average communication time between these collisions for the commencement of resistive flow. Among the various propositions, two are studied in this paper (apart from the classical law). The first model, described by Cattaneo's law [8] instead of classical Fourier's law of conduction, was developed by Lord and Shulman [9]. This law (see Eq. (3.22)) is based on using only one relaxation time by modifying Fourier's law of heat conduction. But Cattaneo's law is unable to account for memory effect, which may prevail in some materials, particularly at low temperatures. This leads to believe that for materials with memory, we have to look for another more general constitutive assumption relating the heat flux to the material thermal history. Gurtin and Pipkin [10] first established a general nonlinear theory of heat conduction in rigid materials with memory, for which thermal disturbances propagate with finite speed. They assumed that the response functional, such as entropy, free energy and heat flux, depends on the present value of the temperature and the integrated histories of the temperature and the temperature gradient. The other heat conduction law proposed in this paper is the Gurtin-Pipkin's law [10] (see Eq. (3.24)).

Under both generalized models, thermal disturbances propagate with finite speed, so that the corresponding equations are of hyperbolic types. These generalized theories are more realistic as they consider the second sound effect, that is, the actual occurrence of wave-like heat propagation with finite speed, and they have practical importance, especially in problems involving high heat fluxes and/or small time intervals.

The purpose of the present work is to extend the gradient-dependent plasticity model to include thermal effects through three models of heat conduction. The derivation of the governing equations is done through continuum mechanics and classical plasticity theory, including balance laws and constitutive equations. The second sound phenomenon is introduced into the governing equations to overcome the paradox of infinite speed.

This work is organized as follows. In Section 2, we develop the thermomechanical coupling that represents the model of the workpiece. To demonstrate the salient feature of the gradient-dependent model of plasticity, we consider in Section 3 the one-dimensional counterpart of the thermoplastic model detailed in Section 2. Thus, linear theory is applicable. In Section 4, using the C_0-semigroup theory, we prove the well-posedness of the thermo-gradient-dependent one-dimensional problem derived in the framework of classical Fourier's law. In Section 5, we show that the corresponding semigroup is exponentially stable. **Section 6 is devoted to the presentation of an implicit finite element tool, specifically developed to integrate the derived constitutive equations and to achieve the corresponding numerical simulations.**

Notations and conventions

The list of notations and conventions used in this paper are clarified in the box bellow. Additional notations will be provided when needed.

f_x partial derivative of f with respect to x.

f_{xx} second-order partial derivative of f with respect to x.

$\dot{f}$ derivative of f with respect to time.

$f \cdot g$ inner product.

$f : g$ double contraction product.

δf virtual field of f.

$\mathrm{grad} f$ gradient of f.

$\mathrm{div} f$ divergence of f.

Δf Laplacian of f.

2 Thermodynamic framework

The thermodynamic formulation of the generalized continuum mechanics is a necessary step to establish the well-suited thermomechanical coupling required for realistic structural computations, which represent the ultimate objective of the approach. Few attempts to derive such thermomechanical effects exist in the literature.

For example, in recently developed strain-gradient and gradient-independent theories [3], a thermodynamic framework for gradient models was proposed, but the thermal effects were not addressed. In this regard, this section is devoted to deriving a thermodynamic consistent formulation to address the thermomechanical behavior of materials utilizing the thermodynamic principles.

The principle of virtual power is used to derive the governing micro-force balance equation which, when augmented by the constitutive relations, results in the yield criterion or the plasticity loading surface. Therefore, the principle of virtual power, which is the assertion that given any sub-body V, the virtual power expended on V by materials or bodies exterior to V (i.e. external power) be equal to the virtual power expended within V (i.e. internal power), can be expressed as follows:

$$\int_V \sigma : \delta\dot{\varepsilon}\, dv = \int_V \mathbf{b}\cdot\delta\dot{\mathbf{u}}\, dv + \int_{\partial V} \mathbf{t}\cdot\delta\dot{\mathbf{u}}\, da - \int_V \rho\,\ddot{\mathbf{u}}\cdot\delta\dot{\mathbf{u}}\, dv, \tag{2.1}$$

where σ is the Cauchy stress tensor, ε is the strain tensor, $\mathbf{b}$ is the volume force vector, $\mathbf{u}$ is the displacement vector, $\mathbf{t}$ is the surface traction vector and ρ is the mass density.

Within the framework of small displacements, the strain tensor ε is related to the displacement vector $\mathbf{u}$ by the following equation:

$$\varepsilon = \frac{1}{2}(\text{grad }\mathbf{u} + (\text{grad }\mathbf{u})^T). \tag{2.2}$$

Using Eq. (2.2) and the divergence theorem, Eq. (2.1) can be rewritten in the following form:

$$\int_V (\text{div}\,\sigma + \mathbf{b} - \rho\,\ddot{\mathbf{u}})\cdot\delta\dot{\mathbf{u}}\, dv + \int_{\partial V} (\mathbf{t} - \sigma\cdot\mathbf{n})\cdot\delta\dot{\mathbf{u}}\, da = 0, \tag{2.3}$$

where $\mathbf{n}$ is the outward unit normal to ∂V.

The virtual velocity field $\delta\dot{\mathbf{u}}$ may be arbitrarily specified if and only if

$$\begin{aligned} &\text{div}\,\sigma + \mathbf{b} = \rho\,\ddot{\mathbf{u}}\,, \\ &\mathbf{t} = \sigma\cdot\mathbf{n}\,. \end{aligned} \tag{2.4}$$

Eq.$(2.4)_1$ expresses the local static ($\rho\,\ddot{\mathbf{u}}=\mathbf{0}$) or dynamic ($\rho\,\ddot{\mathbf{u}} \neq \mathbf{0}$) equilibrium or balance force; while Eq.$(2.4)_2$ defines the traction boundary condition.

As explained before, a thermodynamic consistent framework is taken into account here in order to define the constitutive model counterparts, including temperature effect. In this regard, the second law of thermodynamics is used in order to derive the constitutive equations and the first law is considered to derive the generalized heat equation. The two principles of thermodynamics are postulated respectively as follows:

(i) Conservation of energy:

$$\rho\,\dot{e} = \sigma : \dot{\varepsilon} - \text{div}\,\mathbf{q} + \rho\, r. \tag{2.5}$$

(ii) Entropy production (Clausius-Duhem) inequality:

$$\rho\,\dot{S}\,T{+}T\,\mathrm{div}\,\frac{\mathbf{q}}{T}{-}\rho\,r \geq 0, \tag{2.6}$$

where e is the specific internal energy, T is the absolute temperature, r is the heat supply, S is the specific entropy and $\mathbf{q}$ is the heat flux vector.

By introducing the Helmholtz free energy Ψ, such that $\Psi = e - T\,S$, followed by taking the time derivative of this relation and substituting in (2.5) and (2.6), we get the energy equation

$$\rho\,r{-}\rho\,(\dot{\Psi}{+}\dot{T}\,S + T\,\dot{S}){-}\mathrm{div}\,\mathbf{q} + \sigma : \dot{\varepsilon}{=}0, \tag{2.7}$$

and the Clausius-Duhem inequality:

$$\sigma : \dot{\varepsilon}{-}\rho\,\dot{\Psi}{-}\rho\,S\,\dot{T} - \mathbf{q}\cdot\frac{\mathrm{grad}\,T}{T} \geq 0. \tag{2.8}$$

The mechanical behavior is assumed to be elasto-plastic. Consequently, the total strain tensor ε can be additively decomposed in its elastic (thermoelastic) part ε^e and plastic part ε^p:

$$\varepsilon{=}\,\varepsilon^e{+}\varepsilon^p. \tag{2.9}$$

Note that the tensors ε^e and ε^p cannot be directly expressed in terms of displacement $\mathbf{u}$, unlike the total strain tensor ε (see Eq. (2.2)). In the present work, attention is restricted to isotropic hardening in the modeling of the mechanical behavior. Consequently, the plastic flow can be defined by the following form of consistency condition (see Naghdi and Trapp [11]):

$$\begin{cases} f{=}\,(\sigma_{eq}{-}\sigma_s) < 0 & \Rightarrow\ \textit{elastic loading}\ (\dot{\varepsilon}^p{=}0), \\ f = (\sigma_{eq}{-}\sigma_s) = 0\ \textit{and}\ \dot{f}{=}\,(\dot{\sigma}_{eq}{-}\dot{\sigma}_s) < 0 & \Rightarrow\ \text{elastic unloading}\ (\dot{\varepsilon}^p{=}\,0), \\ f = (\sigma_{eq}{-}\sigma_s) = 0\ \text{and}\ \dot{f}{=}\,(\dot{\sigma}_{eq}{-}\dot{\sigma}_s) = 0 & \Rightarrow\ \text{plastic loading}\ (\dot{\varepsilon}^p \neq 0), \end{cases} \tag{2.10}$$

where σ_{eq} is the equivalent stress, which is function of the Cauchy stress tensor σ, while σ_s is the yield stress, which can be viewed as a measure of isotropic hardening. In the present contribution, a thermo-gradient-dependent formulation of plasticity is used to express the yield stress function σ_s. Within this formulation, σ_s is assumed to depend on the equivalent plastic strain measure χ, its Laplacian $\Delta\,\chi$, and the absolute temperature T:

$$\sigma_s := \sigma_s(\chi, \Delta\,\chi, T). \tag{2.11}$$

The equivalent plastic strain rate $\dot{\chi}$ is related to the plastic strain rate $\dot{\varepsilon}^p$ through the normality law:

$$\dot{\varepsilon}^p{=}\dot{\chi}\,\frac{\partial\,f}{\partial\,\sigma}. \tag{2.12}$$

In the current study, elastic, plastic as well as thermal contributions to the material behavior are considered. Consequently, the constitutive variables are functions of the elastic strain tensor ε^e, the absolute temperature T, and the hardening components χ and $\Delta\,\chi$. Hence, within this thermodynamic framework, the Helmholtz free energy Ψ can be expressed as:

$$\Psi := \Psi(\,\varepsilon^e, \chi, \Delta\,\chi, T). \tag{2.13}$$

According to the expression given above for Ψ, the time differentiation of Eq. (2.13) can be expanded in terms of its derivatives with respect to the internal state variables, as follows:

$$\dot{\Psi}=\frac{\partial \Psi}{\partial \varepsilon^e} : \dot{\varepsilon}^e + \frac{\partial \Psi}{\partial T}\dot{T} + \frac{\partial \Psi}{\partial \chi}\dot{\chi} + \frac{\partial \Psi}{\partial \Delta \chi}\Delta\dot{\chi}\,. \tag{2.14}$$

By substituting Eq. (2.14) into Eq. (2.8) and using Eq. (2.9), one obtains the following thermodynamic constraint:

$$(\sigma-\rho\frac{\partial \Psi}{\partial \varepsilon^e}) : \dot{\varepsilon}^e+\sigma : \dot{\varepsilon}^p - \rho\,(S+\frac{\partial \Psi}{\partial T})\dot{T}-\rho\frac{\partial \Psi}{\partial \chi}\dot{\chi} - \rho\frac{\partial \Psi}{\partial \Delta \chi}\,\Delta\,\dot{\chi} - \mathbf{q}\cdot\frac{\mathrm{grad}\,T}{T} \geq 0\,. \tag{2.15}$$

Classical reasoning, starting from the above inequality, leads to the expression of some state equations. Indeed, one may observe that inequality (2.15) holds for all values of time derivatives, while its coefficients are not functions of time derivatives. From this, it comes that the constitutive equations are compatible with the energy equation if they satisfy the following relations:

$$\sigma=\rho\frac{\partial \Psi}{\partial \varepsilon^e}, \qquad S = -\frac{\partial \Psi}{\partial T}. \tag{2.16}$$

Also, we define the thermodynamic forces R and R^g associated with the internal variables χ and $\Delta\chi$, respectively, by the following relations:

$$R= \rho\frac{\partial \Psi}{\partial \chi} \quad ; \quad R^g= \rho\frac{\partial \Psi}{\partial \Delta \chi}\,. \tag{2.17}$$

The yield stress σ_s may be expressed in terms of its initial value σ_y and the thermodynamic forces R and R^g by the following relation:

$$\sigma_s= \sigma_y + R + R^g\,. \tag{2.18}$$

The introduction of Eqs. (2.16) and (2.17) reduces Eq. (2.15) to:

$$\sigma : \dot{\varepsilon}^p - R\,\dot{\chi} - R^g\,\Delta\dot{\chi} - \mathbf{q}\cdot\frac{\mathrm{grad}\,T}{T} \geq 0\,. \tag{2.19}$$

Using Eq. (2.16) again, $\rho\,\dot{\Psi}$ may be reduced to the following expression:

$$\rho\,\dot{\Psi}=\sigma : \dot{\varepsilon}^e - \rho\,S\,\dot{T} + R\,\dot{\chi} + R^g\,\Delta\,\dot{\chi}. \tag{2.20}$$

Then the energy equation (2.7) becomes:

$$\rho\,r+\sigma : \dot{\varepsilon}^p - R\,\dot{\chi}-R^g\,\Delta\,\dot{\chi} - \rho\;T\,\dot{S}-\mathrm{div}\,\mathbf{q}=0. \tag{2.21}$$

We consider a reference configuration, which is in thermal equilibrium and free from stresses. By expanding Ψ as a power series of the independent variables ε^e, T, χ, and $\Delta\chi$, in which only terms of second order or less are kept, we propose the free energy function in the following quadratic form:

$$\rho\,\Psi=\frac{1}{2}\,\varepsilon^e\!: \mathbf{C} : \varepsilon^e - \theta\;\mathbf{K} : \varepsilon^e - \frac{\rho}{2}\frac{C_v}{T_0}\theta^2 + \frac{1}{2}\,h\,\chi^2 - \frac{1}{2}\,h^g\,(\Delta\chi)^2 - \xi_{iso}\,\chi\,\theta - \xi^g_{iso}\,\Delta\chi\;\theta, \tag{2.22}$$

where T_0 is the initial temperature and θ the difference between the absolute temperature T and T_0, i.e.,

$$\theta = T - T_0. \tag{2.23}$$

We assume that $|\theta/T_0| << 1$. C_v is the specific heat, h and h^g stand for the local hardening/softening modulus and the second-order non-local gradient, respectively. ξ_{iso} and ξ^g_{iso} are parameters related to the coupling between isotropic hardening and temperature effects, $\mathbf{C}$ is the fourth-order elasticity tensor, $\mathbf{K}$ is a second-order tensor defined by the following relation:

$$\mathbf{K}{=}\alpha\,\mathbf{C}:\mathbf{I}, \tag{2.24}$$

where α is the thermal expansion parameter and $\mathbf{I}$ is the second-order identity tensor.

Making use of the quadratic form (2.22), we have:

$$\begin{array}{l} \rho\dfrac{\partial^2\Psi}{\partial\theta\,\partial\varepsilon^e}=-\mathbf{K}, \quad \rho\dfrac{\partial^2\Psi}{\partial\theta\,\partial\chi}=-\xi_{iso}, \quad \rho\dfrac{\partial^2\Psi}{\partial\theta\,\partial\Delta\chi}=-\xi^g_{iso}, \\ \dfrac{\partial^2\Psi}{\partial\theta^2}=-\dfrac{C_v}{T_0}, \quad R=\rho\dfrac{\partial\,\Psi}{\partial\chi}=h\,\chi-\xi_{iso}\,\theta, \quad R^g=\rho\dfrac{\partial\,\Psi}{\partial\Delta\chi}=-h^g\,\Delta\chi-\xi^g_{iso}\,\theta. \end{array} \tag{2.25}$$

Using again the expression (2.22) of $\rho\,\Psi$, the constitutive equations (2.16) take the following form:

$$\begin{array}{l} \sigma{=}\,\mathbf{C}:(\varepsilon^e{-}\alpha\,\theta\,\mathbf{I})\,, \\ \rho\,S=\mathbf{K}:\varepsilon^e+\rho\,\dfrac{C_v}{T_0}\theta+\xi_{iso}\,\chi+\xi^g_{iso}\,\Delta\chi\,. \end{array} \tag{2.26}$$

By using Eqs. (2.17), (2.18), (2.22) and Eq.(2.25), the yield stress σ_s can be expressed as follows:

$$\sigma_s{=}\,\sigma_y+h\,\chi-h^g\,\Delta\chi-\xi\,\theta\,, \tag{2.27}$$

where $\xi\;=\;\xi_{iso}{+}\xi^g_{iso}$ **reflects the thermo-mechanical coupling (i.e., the effect of the variation of temperature θ on the evolution of the yield stress σ_s). The material length scale is taken into account in the present model and the corresponding parameter is dependent on the ratio h^g/h. To illustrate this, let us consider a one-dimensional bar of length ℓ that is loaded in pure tension. To simplify the analysis, let us set parameter ξ to 0 (i.e. there is no effect of the variation of temperature on the evolution of the yield stress σ_s). Under this assumption, Eq.** (2.27) **reduces to the following form:**

$$\sigma_s{=}\,\sigma_y+h\,\chi-h^g\,\Delta\chi, \tag{2.28}$$

which admits the following solution:

$$\chi=A\cos(\frac{x}{\ell})+\frac{\sigma_s-\sigma_y}{h}, \tag{2.29}$$

where A is an integration constant, x is the position of the studied point $(0<x<\ell)$**, and**

$$\ell=\sqrt{-\frac{h^g}{h}}. \tag{2.30}$$

Parameters h and h^g must have opposite signs in order that equations (2.29) **and** (2.30) **make sense. Furthermore, ℓ has a dimension of length.**

By using Eq. $(2.16)_2$ and (2.23), the energy balance equation (2.21) can be rewritten as:

$$-\rho\;(\theta+T_0)\,\frac{d}{dt}\left(\frac{\partial\,\Psi}{\partial\,\theta}\right)=\rho\,r{+}\sigma\;:\dot{\varepsilon}^p-R\,\dot{\chi}{-}R^g\,\Delta\,\dot{\chi}{-}\mathrm{div}\,\mathbf{q}. \tag{2.31}$$

The left-hand side term in the above equation can be expanded by the chain rule as:

$$-\rho\ (\theta+T_0)\,(\frac{\partial^2\Psi}{\partial\theta\,\partial\varepsilon^e}\ :\dot{\varepsilon}^e+\frac{\partial^2\Psi}{\partial\theta\,\partial\chi}\dot{\chi}+\frac{\partial^2\Psi}{\partial\theta\,\partial\Delta\chi}\Delta\dot{\chi}+\frac{\partial^2\Psi}{\partial\theta^2}\dot{\theta})=\rho\,r+\sigma:\dot{\varepsilon}^p-R\,\dot{\chi}-R^g\,\Delta\,\dot{\chi}-\mathrm{div}\,\mathbf{q}\,. \tag{2.32}$$

By using Eqs. (2.25), Eq. (2.32) reduces to:

$$(\theta+T_0)\,(\mathbf{K}:\dot{\varepsilon}^e+\frac{\rho\,C_v}{T_0}\dot{\theta}+\xi_{iso}\,\dot{\chi}+\xi^g_{iso}\,\Delta\dot{\chi})=\rho\,r+\sigma:\dot{\varepsilon}^p-h\,\chi\,\dot{\chi}+\xi_{iso}\,\theta\,\dot{\chi}+h^g\,\Delta\chi\,\Delta\dot{\chi}+\xi^g_{iso}\,\theta\,\Delta\dot{\chi}-\mathrm{div}\,\mathbf{q}. \tag{2.33}$$

The studied material is assumed to be plastically incompressible and, hence, the trace of the plastic strain rate $\dot{\varepsilon}^p$ is equal to 0. Consequently, the double contraction product $\mathbf{K}:\dot{\varepsilon}^p$ is equal to 0 and the term $\mathbf{K}:\dot{\varepsilon}^e$ may be replaced by $\mathbf{K}:\dot{\varepsilon}$. Thus, Eq. (2.33) becomes

$$(\theta+T_0)\,(\mathbf{K}:\dot{\varepsilon}+\frac{\rho\,C_v}{T_0}\dot{\theta}+\xi_{iso}\,\dot{\chi}+\xi^g_{iso}\,\Delta\dot{\chi})=\rho\,r+\sigma:\dot{\varepsilon}^p-h\,\chi\,\dot{\chi}+\xi_{iso}\,\theta\,\dot{\chi}+h^g\,\Delta\chi\,\Delta\dot{\chi}+\xi^g_{iso}\,\theta\,\Delta\dot{\chi}-\mathrm{div}\,\mathbf{q}. \tag{2.34}$$

In summary, the thermomechanical problem under study is defined by the strain-displacement relationship (2.2), the equilibrium equation $(2.4)_1$, the traction boundary condition $(2.4)_2$, the consistency condition (2.10), the normality law (2.12), the stress-strain relationship $(2.26)_1$ and the heat equation (2.34).

3 Application to the one-dimensional thermo-gradient-dependent plasticity

To demonstrate the salient feature of the gradient-dependent model of plasticity, we consider the one-dimensional counterpart of the thermoplasticity model detailed in Section 2. All field variables (tensors such as ε and σ or vectors such as $\mathbf{u}$ and $\mathbf{q}$) have one-dimensional spatial dependence on the coordinate x. Consequently, the equations governing the thermomechanical problem reduces to the following scalar equations:

1. The strain-displacement relationship:

$$\varepsilon=\frac{\partial\,u}{\partial\,x}=u_x, \tag{3.1}$$

   which is equivalent to

$$\dot{\varepsilon}=\frac{\partial\,\dot{u}}{\partial\,x}=\dot{u}_x. \tag{3.2}$$

2. The equilibrium equation in the absence of external and body forces:

$$\frac{\partial\,\sigma}{\partial\,x}=\rho\,\ddot{u}. \tag{3.3}$$

3. The consistency condition:

$$\begin{cases} f=(\sigma-\sigma_s(\chi,\chi_{xx},\theta))<0 & \Rightarrow\ \dot{\chi}=0,\\ f=(\sigma-\sigma_s(\chi,\chi_{xx},\theta))=0\ and\ \dot{f}=(\dot{\sigma}-\dot{\sigma}_s(\chi,\chi_{xx},\theta))<0 & \Rightarrow\ \dot{\chi}=0,\\ f=(\sigma-\sigma_s(\chi,\chi_{xx},\theta))=0\ and\ \dot{f}=(\dot{\sigma}-\dot{\sigma}_s(\chi,\chi_{xx},\theta))=0 & \Rightarrow\ \dot{\chi}>0,\end{cases} \tag{3.4}$$

   where $\chi_{xx}=\dfrac{\partial^2\chi}{\partial\,x^2}$.

4. The normality law:

$$\dot{\varepsilon}^p = \dot{\chi}\,\frac{\partial f}{\partial \sigma}. \tag{3.5}$$

Taking into account the fact that $\partial f/\partial \sigma$ is equal to 1 for this particular one-dimensional case, we get:

$$\dot{\varepsilon}^p = \dot{\chi}. \tag{3.6}$$

5. The stress-strain relationship:

$$\sigma = E(\varepsilon - \varepsilon^p - \alpha\theta), \tag{3.7}$$

where E is the Young modulus. Elasticity and thermal expansion are assumed to be linear during the loading. Consequently, the scalars E and α remain constant.

6. The heat equation:

$$(\theta + T_0)(\alpha\, E\,\dot{\varepsilon} + \frac{\rho\, C_v}{T_0}\dot{\theta} + \xi_{iso}\,\dot{\chi} + \xi^g_{iso}\,\dot{\chi}_{xx}) = \sigma\,\dot{\varepsilon}^p - h\,\chi\,\dot{\chi} + \xi_{iso}\,\theta\,\dot{\chi} + h^g\,\chi_{xx}\,\dot{\chi}_{xx} + \xi^g_{iso}\,\theta\,\dot{\chi}_{xx} - q_x, \tag{3.8}$$

where we have neglected the external heat sources (i.e., $\rho r = 0$) to simplify the thermomechanical model.

As stated in Eq. $(3.4)_3$, plastic flow occurs when $\dot{f}$ is equal to 0. As function f depends on σ, χ, χ_{xx} and θ, $(3.4)_3$ becomes equivalent to:

$$\dot{f} = \frac{\partial f}{\partial \sigma}\dot{\sigma} + \frac{\partial f}{\partial \chi}\dot{\chi} + \frac{\partial f}{\partial \chi_{xx}}\dot{\chi}_{xx} + \frac{\partial f}{\partial \theta}\,\dot{\theta} = 0. \tag{3.9}$$

By using Eq. (3.4) and expression (2.27) of σ_s, we can easily obtain the following relations:

$$\begin{array}{l} \dfrac{\partial f}{\partial \sigma} = 1, \quad \dfrac{\partial f}{\partial \chi} = -\dfrac{\partial \sigma_s}{\partial \chi} = -h, \\ \dfrac{\partial f}{\partial \chi_{xx}} = -\dfrac{\partial \sigma_s}{\partial \chi_{xx}} = h^g, \quad \dfrac{\partial f}{\partial \theta} = -\dfrac{\partial \sigma_s}{\partial \theta} = \xi. \end{array} \tag{3.10}$$

The insertion of relations (3.10) into Eq. (3.9) leads to the following expression of $\dot{\sigma}$:

$$\dot{\sigma} = h\,\dot{\chi} - h^g\,\dot{\chi}_{xx} - \xi\,\dot{\theta}. \tag{3.11}$$

Using Eq. (3.6), Eq. (3.11) becomes:

$$\dot{\sigma} = h\,\dot{\varepsilon}^p - h^g\,\dot{\varepsilon}^p_{xx} - \xi\,\dot{\theta}. \tag{3.12}$$

The medium is assumed to be initially stress and plastic strain free. We assume also that h, h^g and ξ are constant during the loading. Then, Eq. (3.12) can be equivalently written as:

$$\sigma = h\,\varepsilon^p - h^g\,\varepsilon^p_{xx} - \xi\,\theta. \tag{3.13}$$

By combining Eqs. (3.1) and (3.7), we can easily derive the expression of the plastic strain ε^p:

$$\varepsilon^p = \frac{\partial u}{\partial x} - \frac{1}{E}\sigma - \alpha\theta. \tag{3.14}$$

Substituting Eq. (3.14) into Eq. (3.13) and using Eq. (3.3), we get:

$$(E{+}h)\,\sigma = E\,h\frac{\partial\,u}{\partial\,x}{-}Eh^g\frac{\partial^3\,u}{\partial\,x^3}{+}\rho h^g\ \frac{\partial\,\ddot{u}}{\partial\,x}{+}\,\alpha Eh^g\ \frac{\partial^2\,\theta}{\partial\,x^2}{-}E(\,\xi+h\,\alpha)\theta. \tag{3.15}$$

Now inserting Eq. (3.15) into Eq. (3.3), we obtain the equation of motion of a thermo-gradient-dependent strain-softening material in one dimensional setting:

$$\varrho\ddot{u}{=}\,h\frac{\partial^2\,u}{\partial\,x^2}{-}h^g\frac{\partial^4\,u}{\partial\,x^4}+\frac{\rho h^g}{E}\ \frac{\partial^2\,\ddot{u}}{\partial\,x^2}{+}\alpha h^g\ \frac{\partial^3\,\theta}{\partial\,x^3}{-}(\,\xi+h\,\alpha)\frac{\partial\,\theta}{\partial\,x}, \tag{3.16}$$

where $\varrho = \dfrac{\rho(E+h)}{E}$. By using Eq. (3.6), the heat equation (3.8) can be expressed as:

$$(\theta+T_0)(\alpha\,E\,\dot{\varepsilon}{+}\frac{\rho\,C_v}{T_0}\dot{\theta}{+}\xi_{iso}\,\dot{\varepsilon}^p{+}\xi^g_{iso}\,\dot{\varepsilon}^p_{xx}) = \sigma\,\dot{\varepsilon}^p-h\,\varepsilon^p\dot{\varepsilon}^p{+}\xi_{iso}\,\theta\,\dot{\varepsilon}^p{+}h^g\,\varepsilon^p_{xx}\,\dot{\varepsilon}^p_{xx}{+}\xi^g_{iso}\,\theta\,\dot{\varepsilon}^p_{xx}{-}q_x. \tag{3.17}$$

The linear form of (3.17) reduces to

$$T_0(\alpha\,E\,\dot{\varepsilon}{+}\frac{\rho C_v}{T_0}\dot{\theta}+\xi_{iso}\,\dot{\varepsilon}^p+\xi^g_{iso}\,\dot{\varepsilon}^p_{xx}) = -q_x. \tag{3.18}$$

Limiting ourselves to the second order partial derivatives of u and of θ, the heat equation (3.18) takes the form

$$E\,(\alpha+\frac{\xi_{iso}}{E+h})\frac{\partial\dot{u}}{\partial x}+\left(\frac{\rho C_v}{T_0}+\frac{\xi_{iso}(\xi-\alpha E)}{E+h}\right)\dot{\theta} = -\frac{1}{T_0}q_x. \tag{3.19}$$

To avoid that the coupled thermomechanical equations (3.16) and (3.19) be an ill-posed problem, one must ensure that the coefficient $\dfrac{\rho C_v}{T_0}+\dfrac{\xi_{iso}(\xi-\alpha E)}{E+h}$ be positive. For this, the condition $\dfrac{\rho C_v}{T_0}+\dfrac{\xi_{iso}\xi}{E+h} > \dfrac{E\xi_{iso}\alpha}{E+h}$ or just $\xi > \alpha E$ must be set.

In the following, we complete (3.16) and (3.19) by three different laws of heat conduction.

`(i)` Classical Fourier's law: we write the heat conduction equation according to the classical Fourier's law,

$$q = -\kappa\frac{\partial\theta}{\partial x}, \tag{3.20}$$

where κ is the coefficient of thermal conductivity. Substituting the divergence of (3.20) into (3.19), we obtain the classical linear governing equations for thermo-gradient-dependent plasticity:

$$\begin{aligned} &\varrho\ddot{u}-h\frac{\partial^2 u}{\partial x^2}+h^g\frac{\partial^4 u}{\partial x^4}-\frac{\rho h^g}{E}\frac{\partial^2\ddot{u}}{\partial x^2}-\alpha h^g\frac{\partial^3\theta}{\partial x^3}+(\xi+h\alpha)\frac{\partial\theta}{\partial x}=0,\\ &(\alpha+\frac{\xi_{iso}}{E+h})E\frac{\partial\dot{u}}{\partial x}+\left(\frac{\rho C_v}{T_0}+\frac{\xi_{iso}(\xi-\alpha E)}{E+h}\right)\dot{\theta}-\frac{\kappa}{T_0}\frac{\partial^2\theta}{\partial x^2}=0. \end{aligned} \tag{3.21}$$

`(ii)` Cattaneo's law: in this case, a modified law of heat conduction, Cattaneo's law [8], i.e.,

$$\tau_0\frac{\partial q}{\partial t}+q = -\kappa\frac{\partial\theta}{\partial x}, \tag{3.22}$$

replaces the classical Fourier's law (3.20). The positive parameter τ_0 is the relaxation time describing the time lag in the response of the heat flux to a gradient in the temperature.

Substituting the divergence of (3.22) into (3.19), we obtain the linear governing equations for thermo-gradient-dependent plasticity under Cattaneo's law (called also with one relaxation time):

$$
\begin{aligned}
\varrho\ddot{u} - h\frac{\partial^2 u}{\partial x^2} + h^g\frac{\partial^4 u}{\partial x^4} - \frac{\rho h^g}{E}\frac{\partial^2 \ddot{u}}{\partial x^2} - \alpha h^g\frac{\partial^3\theta}{\partial x^3} + (\xi + h\alpha)\frac{\partial\theta}{\partial x} &= 0,\\
(\alpha + \frac{\xi_{iso}}{E+h})E(\tau_0\frac{\partial\ddot{u}}{\partial x} + \frac{\partial\dot{u}}{\partial x}) + \left(\frac{\rho C_v}{T_0} + \frac{\xi_{iso}(\xi-\alpha E)}{E+h}\right)(\tau_0\ddot{\theta} + \dot{\theta}) - \frac{\kappa}{T_0}\frac{\partial^2\theta}{\partial x^2} &= 0.
\end{aligned}
\tag{3.23}
$$

By using Cattaneo's law for heat conduction, the heat equation $(3.23)_2$ becomes hyperbolic and automatically eliminates the paradox of infinite speed. It is easy to see that $(3.23)_2$ can be reduced to the classical heat equation $(3.21)_2$ by taking $\tau_0 = 0$.

`(iii)` Gurtin-Pipkin's law: According to Gurtin-Pipkin's theory [10], the linearized constitutive equation of q is given by

$$
q = -\int_0^\infty \kappa(s)\nabla\theta(t-s)ds,
\tag{3.24}
$$

where $\kappa(s)$ is the heat conductivity relaxation kernel.

Substituting the divergence of (3.24) into Eq. (3.19), we obtain the linear governing equations for thermo-gradient-dependent plasticity under Gurtin-Pipkin's law (called also with memory):

$$
\begin{aligned}
\varrho\ddot{u} - h\frac{\partial^2 u}{\partial x^2} + h^g\frac{\partial^4 u}{\partial x^4} - \frac{\rho h^g}{E}\frac{\partial^2 \ddot{u}}{\partial x^2} - \alpha h^g\frac{\partial^3\theta}{\partial x^3} + (\xi + h\alpha)\frac{\partial\theta}{\partial x} &= 0,\\
(\alpha + \frac{\xi_{iso}}{E+h})E\frac{\partial\dot{u}}{\partial x} + \left(\frac{\rho C_v}{T_0} + \frac{\xi_{iso}(\xi-\alpha E)}{E+h}\right)\dot{\theta} - \frac{1}{T_0}\int_0^\infty \kappa(s)\frac{\partial^2\theta}{\partial x^2}(t-s)ds &= 0.
\end{aligned}
\tag{3.25}
$$

The convolution terms, appearing in $(3.25)_2$, introduced by Gurtin-Pipkin's law entail finite propagation speed for thermal disturbances. Eq. (3.24) can be reduced to classical Fourier's law by taking $\kappa(s) = \delta(s)$ (the Dirac distribution at zero). Besides, if we take

$$
\kappa(s) = \frac{1}{\sigma}e^{-\frac{s}{\sigma}}, \qquad \sigma > 0,
$$

and differentiate (3.24) with respect to t, one can easily arrive (formally) at (3.22).

To the field of equations (3.21), (3.23) or (3.25), we add appropriate boundary and initial conditions.

4 Well-posedness of the classical model

In this section, we shall study the well-posedness of the classical system (3.21). Without loss of generality, we assume that the coefficient $\frac{\rho C_v}{T_0} + \frac{\xi_{iso}(\xi-\alpha E)}{E+h}$ is positive and the term $\alpha h^g\frac{\partial^3\theta}{\partial x^3}$ in $(3.21)_1$ is negligible to make the calculations easier. Then, we consider the following system:

$$
\begin{aligned}
\varrho\ddot{u} - \varpi\frac{\partial^2\ddot{u}}{\partial x^2} + h^g\frac{\partial^4 u}{\partial x^4} - h\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial\theta}{\partial x} &= 0, \qquad (x,t)\in(0,\ell)\times\mathbb{R}^+\\
c\dot{\theta} - k\frac{\partial^2\theta}{\partial x^2} + \beta\frac{\partial\dot{u}}{\partial x} &= 0, \qquad (x,t)\in(0,\ell)\times\mathbb{R}^+
\end{aligned}
\tag{4.1}
$$

where

$$\varrho = \frac{\rho(E+h)}{E}, \quad \varpi = \frac{\rho h^g}{E}, \quad \beta = \xi + h\alpha, \quad c = \frac{\beta(\frac{\rho C_v}{T_0} + \frac{\xi_{iso}(\xi-\alpha E)}{E+h})}{(\alpha + \frac{\xi_{iso}}{E+h})E}, \quad k = \frac{\beta\kappa}{(\alpha + \frac{\xi_{iso}}{E+h})ET_0}.$$

The variable $u = u(x,t)$ represents the vertical deflection of the bar of length ℓ with respect to its reference configuration, and $\theta = \theta(x,t)$, accounting for the variation of temperature with respect to its reference value. We study the well-posedness of system (4.1) subject to the initial conditions

$$u(x,0) = u^0(x), \quad u_t(x,0) = v^0(x), \quad \theta(x,0) = \theta^0(x), \qquad x \in (0,\ell). \tag{4.2}$$

We consider clamped boundary conditions for u and Dirichlet boundary condition for θ, that is:

$$u(x,t) = u_x(x,t) = \theta(x,t) = 0, \quad \text{on} \ \ x = 0,\ell, \quad t > 0. \tag{4.3}$$

We now transform the initial-boundary-value problem given by equations (4.1)-(4.3) to an abstract problem on a suitable Hilbert space. Well-posedness is then obtained by using semigroup theory. Putting $\dot{u} = v$, (4.1) becomes

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \varrho - \varpi D^2 & 0 \\ 0 & 0 & c \end{pmatrix} \frac{d}{dt} \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ (h - h^g D^2)D^2 & 0 & -\beta D \\ 0 & -\beta D & kD^2 \end{pmatrix} \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}, \tag{4.4}$$

where $D^k = \dfrac{\partial^k}{\partial x^k}$. Set

$$\mathscr{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varrho - \varpi D^2 & 0 \\ 0 & 0 & c \end{pmatrix}, \qquad \mathscr{B} = \begin{pmatrix} 0 & 1 & 0 \\ (h - h^g D^2)D^2 & 0 & -\beta D \\ 0 & -\beta D & kD^2 \end{pmatrix}. \tag{4.5}$$

Then Eqs. (4.4) can be written into Sobolev equation

$$\mathscr{A}\frac{d\mathcal{U}(t)}{dt} = \mathscr{B}\mathcal{U}(t), \quad \mathcal{U}(0) = \mathcal{U}_0, \tag{4.6}$$

where

$$\mathcal{U} = (u, v, \theta), \qquad \mathcal{U}_0 = (u^0, v^0, \theta^0). \tag{4.7}$$

Put

$$\begin{aligned}
\mathscr{H} &= H_0^2(\Omega) \times L^2(\Omega) \times L^2(\Omega), \\
\mathscr{D}(\mathscr{A}) &= H_0^2(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega), \\
\mathscr{D}(\mathscr{A}^{1/2}) &= H_0^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) = \mathscr{K}, \\
\mathscr{D}(\mathscr{B}) &= (H^4(\Omega) \cap H_0^2(\Omega)) \times H_0^2(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega)),
\end{aligned}$$

where $\Omega = (0,\ell)$ and $\mathscr{H}$ and $\mathscr{K}$ are Hilbert spaces.

We define the inner products of $\mathscr{H}$ and $\mathscr{K}$ as follows:

$$\left(\begin{pmatrix} u \\ v \\ \theta \end{pmatrix}, \begin{pmatrix} u' \\ v' \\ \theta' \end{pmatrix}\right)_{\mathscr{H}} = h^g(D^2u, D^2u')_{L^2(\Omega)} + h(Du, Du')_{L^2(\Omega)} + (v, v')_{L^2(\Omega)} + (\theta, \theta')_{L^2(\Omega)},$$

$$\left(\begin{pmatrix} u \\ v \\ \theta \end{pmatrix}, \begin{pmatrix} u' \\ v' \\ \theta' \end{pmatrix}\right)_{\mathscr{K}} = h^g(D^2u, D^2u')_{L^2(\Omega)} + h(Du, Du')_{L^2(\Omega)} + (Dv, Dv')_{L^2(\Omega)} + (v, v')_{L^2(\Omega)} + (\theta, \theta')_{L^2(\Omega)}.$$

The following theorem is proved by using the same argument used in [12] in the proof of Theorem 11.11.

Theorem 4.1. *The problem (4.6) is well-posed and is governed by a C_0-contraction semigroup on $\mathscr{K}$.*

Proof. According to Goldstein [12] (see Theorem 11.11), it is sufficient to prove that $\mathscr{A}$ is a positive and self-adjoint operator on $\mathscr{H}$, that 0 belongs to the resolvent set of $\mathscr{A}$, i.e., $0 \in \rho(\mathscr{A})$, that $\mathscr{D}(\mathscr{A}) \supset \mathscr{D}(\mathscr{B})$ and that $\mathscr{B}$ in a maximal dissipative operator on $\mathscr{H}$. Since it is easy and classic to prove that $0 \in \rho(\mathscr{A})$ and that $\mathscr{D}(\mathscr{A}) \supset \mathscr{D}(\mathscr{B})$, we will only prove that $\mathscr{A}$ is a positive and self-adjoint operator on $\mathscr{H}$. For all $\mathcal{U} = (u, v, \theta) \in \mathscr{D}(\mathscr{A})$, we have

$$\begin{aligned}(\mathscr{A}\mathcal{U}, \mathcal{U})_{\mathscr{H}} &= \left(\begin{pmatrix} u \\ (\varrho - \varpi D^2)v \\ c\theta \end{pmatrix}, \begin{pmatrix} u \\ v \\ \theta \end{pmatrix}\right)_{\mathscr{H}} \\ &= h^g\|D^2u\|^2_{L^2(\Omega)} + h\|Du\|^2_{L^2(\Omega)} + \varrho\|v\|^2_{L^2(\Omega)} + \varpi\|Dv\|^2_{L^2(\Omega)} + c\|\theta\|^2_{L^2(\Omega)} \geq 0,\end{aligned} \tag{4.8}$$

which shows that $\mathscr{A}$ is a positive operator on $\mathscr{H}$. To obtain

$$(\mathscr{A}\mathcal{U}, \mathcal{U}')_{\mathscr{H}} = (\mathcal{U}, \mathscr{A}\mathcal{U}')_{\mathscr{H}}$$

we observe that

$$\begin{aligned}((\varrho - \varpi D^2)v, v') &= \varrho(v, v')_{L^2(\Omega)} - \varpi(D^2v, v')_{L^2(\Omega)} \\ &= \varrho(v, v')_{L^2(\Omega)} - \varpi(v, D^2v')_{L^2(\Omega)} \\ &= (v, (\varrho - \varpi D^2)v')_{L^2(\Omega)}.\end{aligned}$$

Hence the operator $\mathscr{A}$ is self adjoint on $\mathscr{H}$.

To prove that $\mathscr{B}$ is a maximal dissipative operator on $\mathscr{H}$, we have for all $\mathcal{U}=(u,v,\theta)\in\mathscr{D}(\mathscr{B})$

$$
\begin{aligned}
(\mathscr{B}\mathcal{U},\mathcal{U})_{\mathscr{H}} &= \left(\begin{pmatrix} v \\ (h-h^g D^2)D^2u-\beta D\theta \\ -\beta Dv+kD^2\theta \end{pmatrix},\begin{pmatrix} u \\ v \\ \theta \end{pmatrix}\right)_{\mathscr{H}} \\
&= h^g(D^2v,D^2u)+h(Dv,Du)+((h-h^gD^2)D^2u-\beta D\theta,v)+(-\beta Dv+kD^2\theta,\theta) \\
&= h^g(D^2v,D^2u)+h(Dv,Du)-h(Du,Dv)-h^g(D^2u,D^2v)-\beta(D\theta,v)+\beta(v,D\theta)-k(D\theta,D\theta) \\
&= 2i\Im m\Big(h^g(D^2v,D^2u)+h(Dv,Du)+\beta(v,D\theta)\Big)-k\|D\theta\|^2
\end{aligned}
\tag{4.9}
$$

from which it follows that

$$
\Re e(\mathscr{B}\mathcal{U},\mathcal{U})_{\mathscr{H}}=-k\|D\theta\|^2, \tag{4.10}
$$

which proves that $\mathscr{B}$ is a dissipative operator on $\mathscr{H}$.

To show that $\mathrm{Range}(I-\mathscr{B})=\mathscr{H}$, we will prove the existence of a vector of functions $(u,v,\theta)\in\mathscr{D}(\mathscr{B})$ satisfying

$$
\begin{pmatrix} u \\ v \\ \theta \end{pmatrix}-\mathscr{B}\begin{pmatrix} u \\ v \\ \theta \end{pmatrix}=\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}\in\mathscr{H} \tag{4.11}
$$

which in terms of the components gives

$$
\begin{aligned}
u-v&=f_1,\\
v-(h-h^gD^2)D^2u+\beta D\theta&=f_2,\\
\theta+\beta Dv-kD^2\theta&=f_3.
\end{aligned}
$$

Inserting the first equation into the others, we have

$$
\begin{aligned}
u-(h-h^gD^2)D^2u+\beta D\theta&=f_1+f_2,\\
\theta+\beta Du-kD^2\theta&=f_3+\beta Df_1,\\
u=Du=\theta&=0 \quad on\ x=0,\ell.
\end{aligned}
\tag{4.12}
$$

To solve (4.12), we consider the following bilinear form defined on $H_0^2(\Omega)\times H_0^1(\Omega)\times H_0^1(\Omega)$ by

$$
\begin{aligned}
\mathcal{F}\left(\begin{pmatrix} u \\ \theta \end{pmatrix},\begin{pmatrix} \phi \\ \varphi \end{pmatrix}\right) &= (u,\phi)+h(Du,D\phi)+h^g(D^2u,D^2\phi)+\beta(D\theta,\phi)+(\theta,\varphi)+k(D\theta,D\varphi) \\
&\quad+\beta(Du,\varphi).
\end{aligned}
$$

We will show that the following variational equation

$$
\mathcal{F}\left(\begin{pmatrix} u \\ \theta \end{pmatrix},\begin{pmatrix} \phi \\ \varphi \end{pmatrix}\right)=(f_1+f_2,\phi)+(f_3+\beta Df_1,\varphi), \tag{4.13}
$$

has a unique solution $(u,\theta)\in H_0^2(\Omega)\times H_0^1(\Omega)$ for any $(\phi,\varphi)\in H_0^2(\Omega)\times H_0^1(\Omega)$. The bilinear form $\mathcal{F}$ is coercive, because

$$\begin{aligned}\mathcal{F}\left(\begin{pmatrix} u\\ \theta\end{pmatrix},\begin{pmatrix} u\\ \theta\end{pmatrix}\right) &= \|u\|^2+h\|Du\|^2+h^g\|D^2u\|^2+\beta(D\theta,u)+\|\theta\|^2+k\|D\theta\|^2-\beta(u,D\theta)\\ &\geq C(\|u\|^2+\|Du\|^2+\|D^2u\|^2+\|\theta\|^2+\|D\theta\|^2).\end{aligned}$$

According to Lax–Milgram theorem, there exists a unique solution of (4.13) satisfying

$$(u,\theta)\in H_0^2(\Omega)\times H_0^1(\Omega).$$

If we put $\phi=0$ in (4.13), then

$$(\theta,\varphi)+k(D\theta,D\varphi)+\beta(Du,\varphi)=(f_3+\beta Df_1,\varphi),\quad \forall\varphi\in H_0^1(\Omega)$$

which implies that θ is a weak solution of the equation

$$\begin{aligned}\theta-kD^2\theta&=f_3+\beta Df_1-\beta Du,\quad \text{in }\Omega\\ \theta&=0,\qquad \text{at } x=0,\ell.\end{aligned}$$

Therefore, $\theta\in H^2(\Omega)\times H_0^1(\Omega)$.

If we put $\varphi=0$ in (4.13) then

$$(u,\phi)+h(Du,D\phi)+h^g(D^2u,D^2\phi)+\beta(D\theta,\phi)=(f_1+f_2,\phi),\quad \forall\phi\in H_0^1(\Omega)$$

which also implies that u is a weak solution to the problem

$$\begin{aligned}u-hD^2u+h^gD^4u&=f_1+f_2-\beta D\theta,\quad \text{in }\Omega\\ u=Du&=0,\qquad \text{at } x=0,\ell.\end{aligned}$$

Therefore $u\in H^4(\Omega)\times H_0^2(\Omega)$. Combining these facts, we conclude that Range$(I-\mathscr{B})=\mathscr{H}$. □

5 Exponential decay of the weak solution

In this section we study the exponential stability of the solution to problem (4.1)-(4.3). It is certainly an interesting problem to determine whether the thermal dissipation under Fourier's law is strong enough by itself to induce exponential stability of this kind of system. We shall prove this in the affirmative. Our main tools are Prüss [13] and Huang [14] results on the stability of semigroups. We can find them in the book by Liu and Zheng [15]. We will use the following result:

Theorem 5.1. *Let $T(t)=e^{t\mathcal{A}}$ a C_0-semigroup of contractions on a Hilbert space $\mathscr{K}$, with infinitesimal generator $\mathcal{A}$ with resolvent set $\rho(\mathcal{A})$. $T(t)$ is exponentially stable if and only if ,*

$$\rho(\mathcal{A})\supseteq\{i\lambda;\ \lambda\in\mathbb{R}\}\equiv i\mathbb{R} \tag{5.1}$$

and

$$\limsup_{|\lambda|\to\infty} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathscr{L}(\mathscr{K})} < \infty, \tag{5.2}$$

where the expression $\|\cdot\|_{\mathscr{L}(\mathscr{K})}$ denotes the norm in the space of continuous linear functions in $\mathscr{K}$.

In view of Theorem 5.1, we prove the following Lemmas.

Lemma 5.1. *The operator $\mathscr{A}\mathscr{B}^{-1} : \mathscr{K} \to \mathscr{K}$ is compact.*

Proof. Since $(\varrho - \varpi D^2)^{-1}$ is a surjective map from $L^2(\Omega)$ into $H^1_0(\Omega)$; $\mathscr{D}(\mathscr{A}^{-1}\mathscr{B}) = \mathscr{D}(\mathscr{B})$. Let us consider (F_n) a bounded sequence in $\mathscr{K}$ and $(\mathcal{U}_n)$ the sequence in $\mathscr{D}(\mathscr{A}^{-1}\mathscr{B})$ such that $F_n = \mathscr{A}^{-1}\mathscr{B}\mathcal{U}_n$, $\mathcal{U}_n = (u_n, v_n, \theta_n)$. Since $\mathscr{A}\mathscr{B}^{-1} \in \mathscr{L}(\mathscr{K})$, there exists a positive constant C such that

$$\|\mathcal{U}_n\|_{\mathscr{K}} + \|\mathscr{A}^{-1}\mathscr{B}\mathcal{U}_n\|_{\mathscr{K}} \le C, \qquad \text{for all } n \in \mathbb{N}. \tag{5.3}$$

From (5.3), we conclude that (u_n, v_n, θ_n) is bounded in $\mathscr{D}(\mathscr{A}^{-1}\mathscr{B})$. Since the embedding of $H^m(0,\pi)$ in $H^j(0,\pi)$, $m > j$, is compact, there exists a subsequence $(u_\nu, v_\nu, \theta_\nu)$ and functions (u, v, θ) such that

$$(u_\nu, v_\nu, \theta_\nu) \to (u, v, \theta) \ \text{ in } \ \mathscr{K},$$

that is, the subsequence $(\mathscr{A}\mathscr{B}^{-1}F_\nu)$ converges in $\mathscr{K}$. $\square$

Lemma 5.2. *The operator $\mathscr{A}^{-1}\mathscr{B}$ satisfies* (5.1)*.*

Proof. We only need to show that there is no point spectrum on the imaginary axis, i.e., $i\mathbb{R} \cap \sigma_p(\mathscr{A}^{-1}\mathscr{B}) = \emptyset$.

Suppose that there exists $\lambda \in \mathbb{R}$, $\lambda \neq 0$, such that $i\lambda$ is in the spectrum of $\mathscr{A}^{-1}\mathscr{B}$. Since $\mathscr{A}\mathscr{B}^{-1}$ is compact, then $i\lambda$ must be an eigenvalue of $\mathscr{A}^{-1}\mathscr{B}$. Therefore, there is a vector $\mathcal{U} \in \mathscr{D}(\mathscr{A}^{-1}\mathscr{B})$, $\mathcal{U} \neq 0$, such that $(i\lambda\mathscr{A} - \mathscr{B})\mathcal{U} = 0$ in $\mathscr{K}$ or equivalently

$$\begin{aligned} i\lambda u - v &= 0,\\ i\lambda(\varrho - \varpi D^2)v + h^g D^4 u - hD^2 u + \beta D\theta &= 0,\\ i\lambda c\theta - kD^2\theta + \beta Dv &= 0. \end{aligned} \tag{5.4}$$

Since $< (i\lambda\mathscr{A} - \mathscr{B})\mathcal{U}, \mathcal{U} >_{\mathscr{K}} = 0$, we have

$$k\int_0^\ell |D\theta|^2 dx = 0,$$

and then $\theta = 0$. By $(5.4)_{1,3}$ we get $u = v = 0$. Thus, we have a contradiction and the proof is complete. $\square$

Lemma 5.3. *The operator $\mathscr{A}^{-1}\mathscr{B}$ satisfies* (5.2)*.*

Proof. It suffices to show that there exists a positive constant C such that for $\lambda \in \mathbb{R}$,

$$\|(i\lambda I - \mathscr{A}^{-1}\mathscr{B})^{-1}\mathcal{U}\|_{\mathscr{K}} \le C\|F\|_{\mathscr{K}}, \quad \text{for all } F \in \mathscr{K}. \tag{5.5}$$

Since $C_0^\infty(\Omega)$ is dense in $\mathscr{K}$, we may assume that $F=(f_1,f_2,f_3)\in C_0^\infty(\Omega)$. Put

$$(i\lambda I-\mathscr{A}^{-1}\mathscr{B})^{-1}F=\mathcal{U}\in\mathscr{D}(\mathscr{A}^{-1}\mathscr{B}),$$

then

$$(i\lambda I-\mathscr{A}^{-1}\mathscr{B})\mathcal{U}=F$$

which implies that

$$(i\lambda\mathscr{A}-\mathscr{B})\mathcal{U}=\mathscr{A}F. \tag{5.6}$$

To prove (5.5), it is sufficient to show that

$$\|\mathcal{U}\|_{H^2(\Omega)\times H^1(\Omega)\times H^1(\Omega)}\le C\|F\|_{\mathscr{K}}. \tag{5.7}$$

We can write (5.6) componentwise as follows:

$$\begin{aligned} i\lambda u-v &= f_1,\\ i\lambda(\varrho-\varpi D^2)v+h^gD^4u-hD^2u+\beta D\theta &= (\varrho-\varpi D^2)f_2,\\ i\lambda c\theta-kD^2\theta+\beta Dv &= cf_3. \end{aligned} \tag{5.8}$$

From Eqs. (4.8) and (4.9), we infer that

$$\Re e<(i\lambda\mathscr{A}-\mathscr{B})\mathcal{U},\mathcal{U}>_{\mathscr{K}}=k\int_0^\ell|D\theta|^2dx$$

and then from (5.6), we conclude that

$$\int_0^\ell|D\theta|^2dx\le C\|F\|_{\mathscr{K}}\|\mathcal{U}\|_{\mathscr{K}}, \tag{5.9}$$

for a positive constant C. Multiplying $(5.8)_2$ by u and using $(5.8)_1$ we obtain

$$\begin{aligned} h^g\int_0^\ell|D^2u|^2dx+h\int_0^\ell|Du|^2dx &= \varrho\int_0^\ell|v|^2dx+\varpi\int_0^\ell|Dv|^2dx+\beta\int_0^\ell\theta D\bar{u}dx+\varrho\int_0^\ell v\bar{f}_1dx\\ &- \varpi\int_0^\ell v\overline{D^2f_1}dx+\varrho\int_0^\ell f_2\bar{u}dx-\varpi\int_0^\ell D^2f_2\bar{u}dx, \end{aligned} \tag{5.10}$$

which, using (5.9) implies that

$$h^g\int_0^\ell|D^2u|^2dx+\frac{h}{2}\int_0^\ell|Du|^2dx\le\varpi\int_0^\ell|Dv|^2dx+\varrho\int_0^\ell|v|^2dx+C\|F\|_{\mathscr{K}}\|\mathcal{U}\|_{\mathscr{K}}, \tag{5.11}$$

for a positive constant C. Multiplying now $(5.8)_2$ by v and using $(5.8)_1$ we obtain

$$\begin{aligned} &i\lambda\Big(\varrho\int_0^\ell|v|^2dx+\varpi\int_0^\ell|Dv|^2dx+h^g\int_0^\ell|D^2u|^2dx+h\int_0^\ell|Du|^2dx\Big)\\ =\ &-\beta\int_0^\ell D\theta\bar{v}dx+\varrho\int_0^\ell f_2\bar{v}dx+\varpi\int_0^\ell Df_2D\bar{v}dx+h^g\int_0^\ell D^2uD^2\bar{f}_1dx+h\int_0^\ell DuD\bar{f}_1dx. \end{aligned} \tag{5.12}$$

Taking the imaginary part of (5.12), we have

$$h^g\int_0^\ell |D^2u|^2dx + h\int_0^\ell |Du|^2dx + \varpi\int_0^\ell |Dv|^2dx + \frac{\varrho}{2}\int_0^\ell |v|^2dx \le C\|F\|_{\mathscr{K}}\|\mathcal{U}\|_{\mathscr{K}}, \tag{5.13}$$

for a positive constant C. Multiplying (5.13) by 3 and summing up with (5.11), we get

$$4h^g\int_0^\ell |D^2u|^2dx + \frac{7h}{2}\int_0^\ell |Du|^2dx + 2\varpi\int_0^\ell |Dv|^2dx + \frac{\varrho}{2}\int_0^\ell |v|^2dx \le C\|F\|_{\mathscr{K}}\|\mathcal{U}\|_{\mathscr{K}}. \tag{5.14}$$

Therefore, combining (5.9) and (5.14), there exists a constant $C > 0$ independent of λ and $F \in \mathscr{K}$ such that (5.7) holds, which implies condition (5.2). Thus, the proof is complete. □

6 Numerical experiments

In this section, we present the numerical scheme that we have implemented in MATLAB in order to obtain a numerical solution to the problem (4.1)-(4.3). Without loss of generality, the numerical approach is developed here for a bar, which is considered as one-dimensional body. To clearly present our numerical approach, let us start our numerical development from (4.1) complemented by the consistency condition (when the loading is elastic-plastic):

$$\sigma - \sigma_s = 0 \Longleftrightarrow E(\varepsilon - \varepsilon^p) - \sigma_y - h\varepsilon^p + h^g\varepsilon^p_{xx} + \xi\theta = 0. \tag{6.1}$$

The weak form of (4.1) together with (6.1) can be written as:

$$\begin{aligned}
\int_0^\ell \delta u\Big(\varrho\ddot{u} - \varpi\frac{\partial^2\ddot{u}}{\partial x^2} + h^g\frac{\partial^4 u}{\partial x^4} - h\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial\theta}{\partial x}\Big)dx = 0,\\
\int_0^\ell \delta\theta\Big(c\dot{\theta} - k\frac{\partial^2\theta}{\partial x^2} + \beta\frac{\partial\dot{u}}{\partial x}\Big)dx = 0,\\
\int_0^\ell \delta\varepsilon^p\Big(E(\varepsilon - \varepsilon^p) - \sigma_y - h\varepsilon^p + h^g\varepsilon^p_{xx} + \xi\theta\Big)dx = 0,
\end{aligned} \tag{6.2}$$

where the δ-symbol denotes the variation of the corresponding quantity. With aid of the divergence theorem and taking into account the boundary conditions (4.3), Eqs. (6.2) can be transformed as:

$$\begin{aligned}
\varrho\int_0^\ell \delta u\ddot{u}dx + \int_0^\ell \delta\varepsilon\Big(\varpi\frac{\partial\ddot{u}}{\partial x} - h^g\frac{\partial^3 u}{\partial x^3} + h\frac{\partial u}{\partial x} - \beta\theta\Big)dx = 0,\\
k\int_0^\ell \delta(\frac{\partial\theta}{\partial x})^2dx + \int_0^\ell \delta\theta\Big(c\dot{\theta} + \beta\frac{\partial\dot{u}}{\partial x}\Big)dx = 0,\\
\int_0^\ell \delta\varepsilon^p\Big(E(\varepsilon - \varepsilon^p) - \sigma_y - h\varepsilon^p + h^g\varepsilon^p_{xx} + \xi\theta\Big)dx = 0.
\end{aligned} \tag{6.3}$$

The main unknowns of the problem (6.3) are the displacement u, the plastic deformation ε^p, its Laplacian ε^p_{xx} and the temperature variation θ. The studied bar is discretized into n finite elements. Each element is defined by two nodes (one at each end). In this case, a linear interpolation is used for fields u and θ and a Hermitian interpolation is employed for ε^p. The continuous displacement field u and temperature variation θ are discretized as follows:

$$u = \mathbf{H}\,\mathbf{a}, \qquad \theta = \mathbf{H}\,\mathbf{b} \tag{6.4}$$

where $\mathbf{H}$ is a matrix containing the linear interpolation polynomials and $\mathbf{a}$ (resp. $\mathbf{b}$) is the nodal displacement (resp. temperature variation) vector defined as follows:

$$\begin{aligned} \mathbf{a} &= \{a_1, a_2, \cdots, a_n, a_{n+1}\} \text{ with } a_1 = a_{n+1} = 0, \\ \mathbf{b} &= \{b_1, b_2, \cdots, b_n, b_{n+1}\} \text{ with } b_1 = b_{n+1} = 0. \end{aligned} \tag{6.5}$$

The strain field ε is related to the nodal displacement vector $\mathbf{a}$ by the following expression:

$$\varepsilon = \mathbf{H}_x \mathbf{a}. \tag{6.6}$$

Similarly, the plastic deformation ε^p can be discretized as follows:

$$\varepsilon^p = \mathbf{h}\,\mathbf{d} \tag{6.7}$$

with $\mathbf{h}$ is a matrix containing the Hermitian interpolation polynomials and Ψ is defined as follows:

$$\mathbf{d} = \{d_1, d_2, \cdots, d_n, d_{n+1}\} \text{ with } d_1 = d_{n+1} = 0.$$

By following the same approach developed in [20], Eq. (6.3) can be transformed to the following matrix form (after some mathematical developments):

$$\begin{aligned} \mathbf{K_{aa}a} + \mathbf{K_{ad}d} + \mathbf{M\ddot{a}} &= 0, \\ \mathbf{K_{bb}b} + \mathbf{C_a\dot{a}} + \mathbf{C_b\dot{b}} &= 0, \\ \mathbf{K_{ad}^T a} + \mathbf{K_{dd}d} + \mathbf{K_{db}b} + \mathbf{F_d} &= 0, \end{aligned} \tag{6.8}$$

where

$$\begin{aligned} &\mathbf{K_{aa}} = E\int_0^\ell \mathbf{H}_x^T\mathbf{H}_x dx, \qquad \mathbf{K_{ad}} = -E\int_0^\ell \mathbf{H}_x^T\mathbf{h}^T dx, \qquad \mathbf{M} = \varrho\int_0^\ell \mathbf{H}^T\mathbf{H}dx, \\ &\mathbf{K_{bb}} = k\int_0^\ell \mathbf{H}_x^T\mathbf{H}_x dx, \qquad \mathbf{C_a} = \beta\int_0^\ell \mathbf{H}_x^T\mathbf{H}dx, \qquad \mathbf{C_b} = c\int_0^\ell \mathbf{H}^T\mathbf{H}dx, \\ &\mathbf{K_{dd}} = \int_0^\ell [(h+E)\mathbf{h}\mathbf{h}^T - h^g\mathbf{h}\mathbf{h}_{xx}^T]dx, \qquad \mathbf{K_{db}} = -\xi\int_0^\ell \mathbf{h}\mathbf{H}^T dx, \qquad \mathbf{F_d} = \sigma_y\int_0^\ell \mathbf{h}dx. \end{aligned} \tag{6.9}$$

To determine the evolution of the different unknowns (namely $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{d}$) during the loading history, the time interval is subdivided into several time increments which are typically noted $I_j = [t_j, t_{j+1}]$. Over each time increment, we assume that $\mathbf{a}(t_j)$, $\mathbf{b}(t_j)$ and $\mathbf{d}(t_j)$ are known and the goal is to compute $\mathbf{a}(t_{j+1})$, $\mathbf{b}(t_{j+1})$ and $\mathbf{d}(t_{j+1})$. Velocities $\dot{\mathbf{a}}$, $\ddot{\mathbf{a}}$ and $\dot{\mathbf{b}}$ are approximated as follows:

$$\begin{aligned} \dot{\mathbf{a}} &= \frac{1}{\Delta t}(\mathbf{a}(t_j) - \mathbf{a}(t_{j-1})), \\ \ddot{\mathbf{a}} &= \frac{1}{(\Delta t)^2}(\mathbf{a}(t_j) - 2\mathbf{a}(t_{j-1}) + \mathbf{a}(t_{j-2})), \\ \dot{\mathbf{b}} &= \frac{1}{\Delta t}(\mathbf{b}(t_j) - \mathbf{b}(t_{j-1})), \end{aligned} \tag{6.10}$$

where $\Delta t = t_j - t_{j-1} = t_{j-1} - t_{j-2}$. By using Eqs. (6.10), (6.8) can be transformed in the following form:

$$\begin{aligned} \mathbf{a}(t_j) &= (\mathbf{K_{aa}} + \frac{\mathbf{M}}{(\Delta t)^2})^{-1}[-\mathbf{K_{ad}d}(t_j) - \frac{\mathbf{M}}{(\Delta t)^2}(-2\mathbf{a}(t_{j-1}) + \mathbf{a}(t_{j-2}))], \\ \mathbf{b}(t_j) &= -(\mathbf{K_{bb}} + \frac{\mathbf{C_b}}{\Delta t})^{-1}\frac{\mathbf{C_a}}{\Delta t}(\mathbf{a}(t_j) - \mathbf{a}(t_{j-1})) + (\mathbf{K_{bb}} + \frac{\mathbf{C_b}}{\Delta t})^{-1}\frac{\mathbf{C_b}}{\Delta t}\mathbf{b}(t_{j-1}), \\ \mathbf{d}(t_j) &= -\mathbf{K_{dd}^{-1}K_{ad}^T a}(t_j) - \mathbf{K_{dd}^{-1}K_{db}b}(t_j) - \mathbf{K_{dd}^{-1}F_d}. \end{aligned} \tag{6.11}$$

System (6.11) is strongly nonlinear and we have used an implicit iterative scheme to solve it. This iterative scheme is based on the fixed point method. At each iteration k, the following linear system of algebraic equations should be solved:

$$\begin{aligned}
\mathbf{a}(t_{j,k}) &= (\mathbf{K_{aa}} + \frac{\mathbf{M}}{(\Delta t)^2})^{-1}[-\mathbf{K_{ad}}\mathbf{d}(t_{j,k-1}) - \frac{\mathbf{M}}{(\Delta t)^2}(-2\mathbf{a}(t_{j-1}) + \mathbf{a}(t_{j-2}))],\\
\mathbf{b}(t_{j,k}) &= -(\mathbf{K_{bb}} + \frac{\mathbf{C_b}}{\Delta t})^{-1}\frac{\mathbf{C_a}}{\Delta t}(\mathbf{a}(t_{j,k}) - \mathbf{a}(t_{j-1})) + (\mathbf{K_{bb}} + \frac{\mathbf{C_b}}{\Delta t})^{-1}\frac{\mathbf{C_b}}{\Delta t}\mathbf{b}(t_{j-1}),\\
\mathbf{d}(t_{j,k}) &= -\mathbf{K_{dd}^{-1}}\mathbf{K_{ad}^{T}}\mathbf{a}(t_{j,k}) - \mathbf{K_{dd}^{-1}}\mathbf{K_{db}}\mathbf{b}(t_{j,k}) - \mathbf{K_{dd}^{-1}}\mathbf{F_d}.
\end{aligned}$$

where

$$\mathbf{a}(t_{j,0}) = \mathbf{a}(t_{j-1}), \qquad \mathbf{b}(t_{j,0}) = \mathbf{b}(t_{j-1}), \qquad \mathbf{d}(t_{j,0}) = \mathbf{d}(t_{j-1}).$$

The last linear system has a unique solution, since the coefficient matrices have non-zero determinant. The iterative procedure converges when the following conditions are fulfilled:

$$\|\mathbf{a}(t_{j,k}) - \mathbf{a}(t_{j,k-1})\| \le 10^{-7}, \qquad \|\mathbf{b}(t_{j,k}) - \mathbf{b}(t_{j,k-1})\| \le 10^{-7}, \qquad \|\mathbf{d}(t_{j,k}) - \mathbf{d}(t_{j,k-1})\| \le 10^{-7}.$$

We describe in what follows the results of some numerical simulations. The length ℓ of this bar is set to 1 m. In all simulations, the bar is discretized into 100 elements which have exactly the same initial length (0.01 m). The adopted time increment Δt is set to $10^{-4}s$. The material parameters corresponding to AK steel 321 material are given below [21]:

Table 1: The values of material parameters.

$\rho = 9010$ Kg/m^3	$E = 193$ GPa	$h = 592$ MPa
$\kappa = 386$ W/(m K)	$h^g = -1$ MPa m^2	$\alpha = 16 \times 10^{-16}$ K^{-1}
$\xi = 2$ MPa K	$\xi_{iso} = 1$ MPa K	$C_v = 500$ J/kg K
$T_0 = 293$ K	$\sigma_y = 0$ MPa	$\varpi = 4.668 \times 10^{-9}$ kg/(N.m^2)

The initial temperature and the initial displacement are given by the following equations:

$$u_0(x) = 0.01x(\ell - x), \qquad u_1(x) = 0, \qquad \theta_0(x) = 0, \qquad \text{for all } x \in [0, \ell].$$

The evolutions of temperature θ and displacement u for the node at the middle of the bar are given in Figures 1 and 2. These results show that both θ and u oscillate at the beginning of the loading and tend to zero very quickly. This result is expectable considering the constitutive equations of the model and the adopted boundary conditions.

In Figures 3 and 4, the temperature and displacement fields are plotted at different times. The results of these Figures confirm and generalize the trends obtained in Figures 1 and 2. Indeed, we observe from these Figures the fast decay of the evolution, which confirms the results obtained in Section 5 about the exponential decay of the solution.

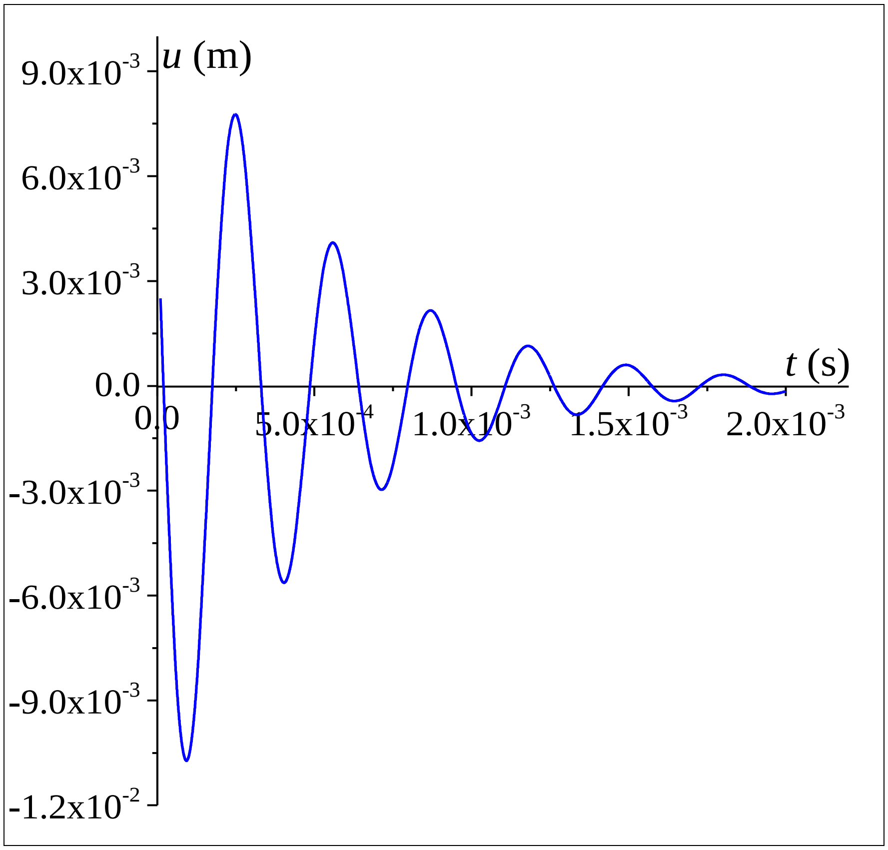

Figure 1: The time evolution of the displacement at the middle of the bar.

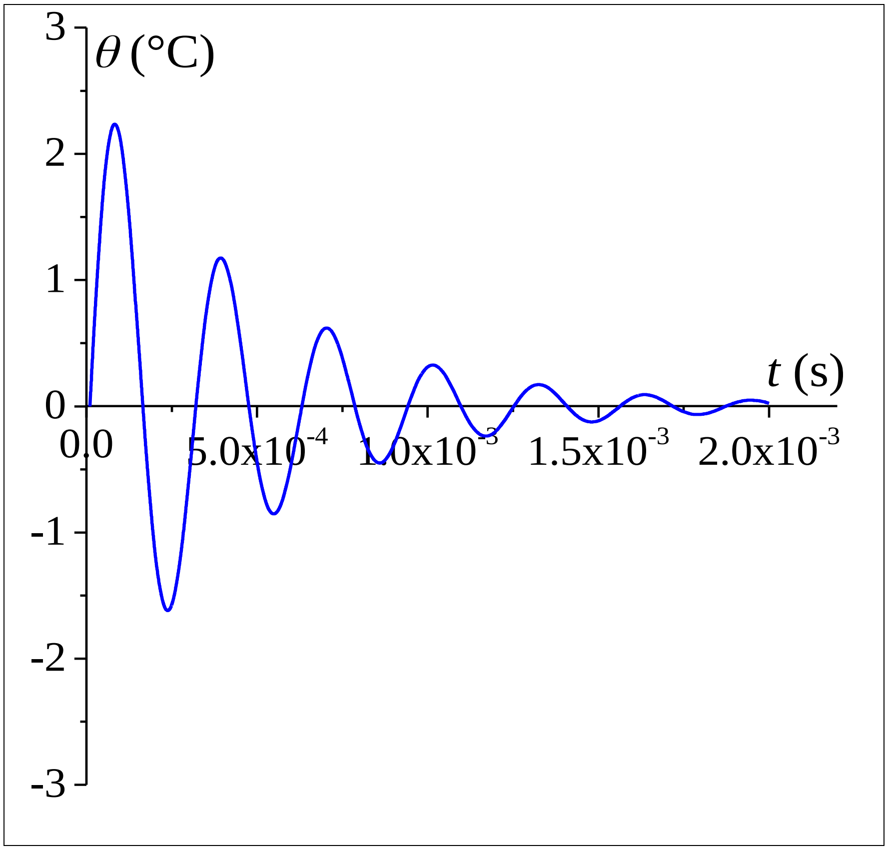

Figure 2: The time evolution of the temperature at the middle of the bar.

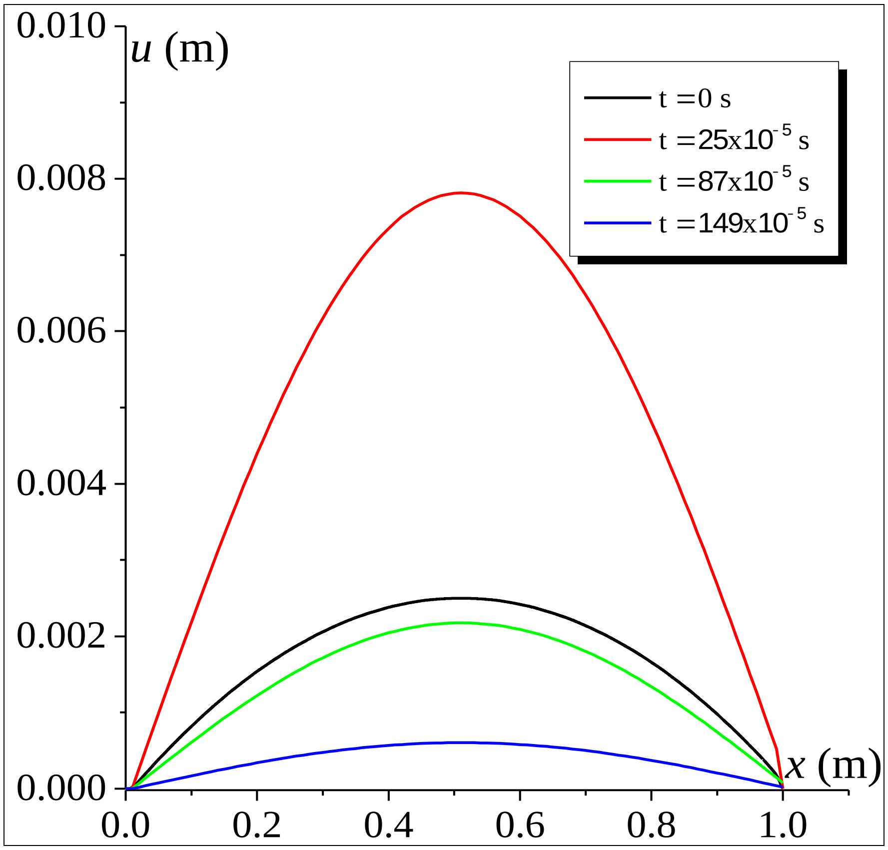

Figure 3: The displacement as a function of x at different times.

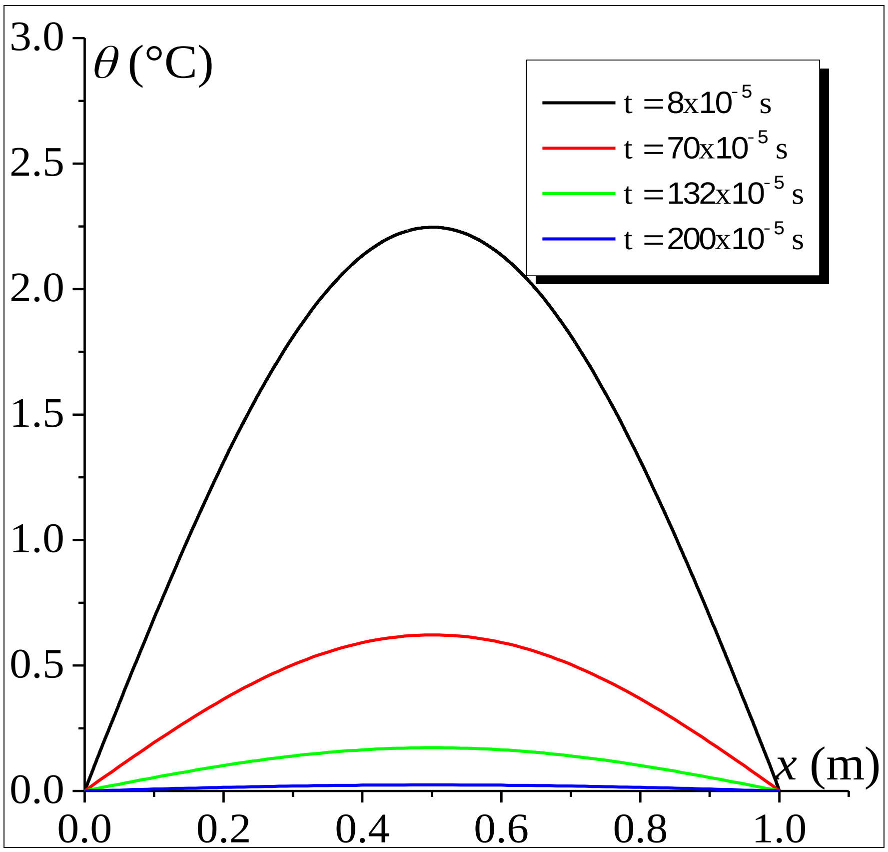

Figure 4: The temperature as a function of x at different times.

7 Conclusion

We summarize the obtained results as follows:

(i) The model presented in this paper can be considered as a feasible thermodynamic approach that enables one to derive various coupled thermo-gradient-dependent theories of plasticity, within different realistic heat conduction laws by introducing simplifying assumptions. By comparison with other gradient-dependent plasticity models [16, 17, 18, 19], the model proposed in this paper is more reasonable in predicting the propagation of thermal and elastic-plastic waves. It can be seen in the above equations (see e.g. Eq. (4.10)) that the only dissipative counterpart is due to the presence of temperature in the model. This work, which has not been obtained in any reference yet, represents a first step towards understanding the fundamental limits of intrinsic thermal dissipations in gradient-dependent plastic materials.

(ii) By means of semigroup theory, the well-posedness of the thermo-gradient-dependent plastic one-dimensional problem was proved and its exponential stability was derived. The well-posedness result proves that in the motion following any sufficiently small change in the external system, the solution of the initial-boundary value problem is everywhere arbitrarily small in magnitude. The exponential stability result, from a mathematical point of view, means that there exist positive constants M and ω such that $\|\mathcal{U}(t)\| \leq Me^{-\omega t}\|\mathcal{U}(0)\|$, where $\mathcal{U}(t)$ and $\mathcal{U}(0)$ are defined by (4.7). This means that the dissipation induced by the thermal effect (see Eq. (4.10)) is strong enough to produce a uniform decay of the solution. From a mechanical point of view, it implies that if we consider a thermoelastic perturbation, then after a small period of time, the perturbations are so small that they can be neglected.

(iii) **The one-dimensional problem of the basic equations has been solved numerically for a particular material and for special initial and boundary conditions involving a periodic mechanical regime and an exponentially decreasing regime. Functions of practical interest have been obtained and the numerical results have been plotted and discussed. In particular, we show the fast decay of the solution, which confirms the results obtained in Section 5. We remark from Figures 1 and 2, that the studied functions have non-zero value (although it may be very small). This is due to the fact that our model is based on the classical Fourier's law, where an infinite speed of propagation is inherent. We expect that under a second sound law, such as Cattaneo's law or Gurtin-Pipkin's law, the corresponding functions vanish identically outside a bounded region of space surrounding the heating source at a distance from it. This may be studied in some future paper.**

The results presented in this paper should prove useful for researchers in materials science, designers of new materials, low-temperature physicists, as well as for those working on the development of higher-order gradient theories. In particular, the numerical schemes will be useful in simulation and identification studies to predict and better understand the structural and thermal responses of such systems.

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