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# A new system formulation for the tolerance analysis of overconstrained mechanisms

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## Abstract

The goal of tolerance analysis is to verify whether design tolerances enable a mechanism to be functional. The current method consists in computing a probability of failure using Monte Carlo simulation combined with an optimization scheme called at each iteration. This time consuming technique is not appropriate for complex overconstrained systems. This paper proposes a transformation of the current tolerance analysis problem formulation into a parallel system probability assessment problem using the Lagrange dual form of the optimization problem. The number of events being very large, a preliminary selective search algorithm is used to identify the most contributing events to the probability of failure value. The First Order Reliability Method (FORM) for systems is eventually applied to compute the probability of failure at low cost. The proposed method is tested on an overconstrained mechanism modeled in three dimensions. Results are consistent with those obtained with the Monte Carlo simulation and the computing time is significantly reduced.

*Keywords:* tolerance analysis, Lagrange dual, FORM, system reliability, selective search

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## 1. Introduction

Nomenclature	
$P_f$	Functional failure probability
$y_{th}$	Threshold value
$X$	Geometrical deviation = Random variable
$g$	Gap variable = optimization variable
$C_f \geq 0$	Functional condition
$C_{f,dual} \geq 0$	Functional condition in the dual form
$C \leq 0$	Interface constraints
$N_C$	Number of interface constraints
$N_s$	Number of possible situations
$N_{as} \leq N_s$	Number of admissible situations
$N_{ds} \leq N_{as}$	Number of dominant admissible situations

A manufacturing process is not able to provide exactly the same workpieces; indeed, theoretical dimensions of a design product cannot be reached in a repetitive manner (tool wear, operator variability, ...). The mechanism behavior is disturbed by geometrical deviations as well as gaps between different parts of the mechanism. Design tolerances are therefore specified on different features of the mechanism to limit the deviations. Tolerance analysis aims at analyzing the impact of these admissible variations on the mechanism behavior. The main stake is to evaluate a quality level of the product during its design stage. The technique used consists of assessing a probability of failure  $P_f$  of the mechanism of magnitude around  $10^{-6}$  for large series production. This value represents the probability that a functional condition,  $C_f = y_{th} - Y \geq 0$ , is not satisfied, where  $Y$  is a functional characteristic of the mechanism and  $y_{th}$  is a threshold value to not be exceeded.

Tolerance analysis methods must consider the geometrical deviations as random variables whose probabilistic distributions are chosen regarding the manufacturing process [1, 2]. However, gaps between parts or contact points cannot

18 be modeled by aleatory uncertainty. Gaps belong to the parameter uncertainty  
 19 category [3] of the epistemic uncertainty which makes difficult the mechanical  
 20 behavior of this kind of mechanisms to be modeled. Indeed, analyzing iso-  
 21 constrained or overconstrained mechanisms is different. An assembly which  
 22 have only its six degrees of freedom fixed in three dimensions (three degrees of  
 23 freedom in two dimensions) is considered to be an isoconstrained mechanism,  
 24 usually without gaps. On the contrary, an assembly which have more than six  
 25 degrees of freedom fixed is considered as an overconstrained mechanism. Gaps  
 26 allows this kind of mechanism to be assembled although more than six degrees  
 27 of freedom are fixed. Figure 1 shows a simple isoconstrained mechanism in one  
 28 dimension where the functional characteristic  $Y$  must not exceed a specified  
 29 threshold. On such a mechanism, the expression of this characteristic  $Y$  is a  
 30 function only of the dimensions  $x_1$  and  $x_2$ . This kind of problem is well-defined.  
 In contrast, Figure 2 shows two configurations of an overconstrained mechanism.

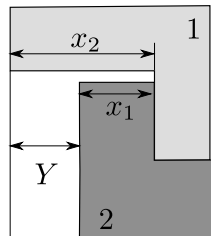


Figure 1: Isostatic mechanism. The functional characteristic  $Y$  can be expressed as a function of variables  $x_1$  and  $x_2$ .

31  
 32 Now the functional characteristic  $Y$  is a function of the random variables and  
 33 of the gap values. However, following the realization of the random variables,  
 34 gap values are depending on the location of the contact point between part 1  
 35 and part 2. In this case the tolerance analysis problem is overconstrained due to  
 36 the multiple possible configurations of gaps. Tolerance analysis methods must  
 37 therefore take into account the worst configurations of gaps to compute the  
 38 probability of failure. This operation is usually performed using an optimiza-  
 39 tion scheme [1]. This particularity turns out to deal with a system probability

40 assessment, the transition from one worst configuration to another leading to  
 an abrupt change of the limit-state surface.

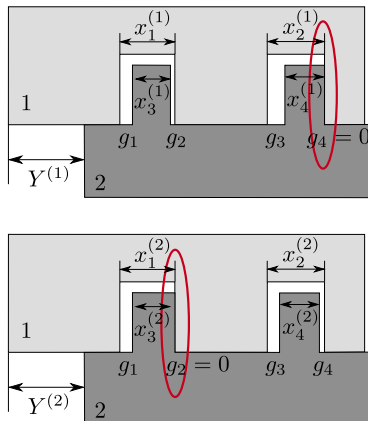


Figure 2: Two worst configurations of gaps in a 1D mechanism. The goal is to find the maximum value of  $Y$ . For two given sets of random variable realizations  $\{\mathbf{x}_i^{(j)}, i = 1, \dots, 4; j = 1, 2\}$ , the worst configuration of gaps changes: contact with the right pin for  $j = 1$  and contact with the left pin for  $j = 2$ .

41

42 For an isoconstrained mechanism, computing the probability of failure is  
 43 simple because the functional condition is only expressed as a function of the  
 44 random variables which describes the geometrical parameters. Classical reli-  
 45 ability methods such as Monte Carlo simulation or variance reduction techniques  
 46 like Importance Sampling [4] can be used to quickly compute the probability  
 47 of functional failure. For complex non explicit applications, surrogate models  
 48 replacing the true functional condition may also be used in simulations in or-  
 49 der to save time. Numerous techniques used in computer experiments exist in  
 50 the literature such as quadratic response surfaces [5, 6], Kriging [7, 8], support  
 51 vector machines [9, 10, 11] and polynomial chaos [12, 13, 14]. Approximation  
 52 methods like FORM [15, 16] can also be performed. All methods are efficient  
 53 provided that the problem has a smooth limit-state surface which is the case  
 54 for an isoconstrained mechanism tolerance analysis problem. For an overcon-  
 55 strained mechanism, these techniques, except the Monte Carlo simulation, can  
 56 no longer be used because of a piecewise limit-state function coming from the

57 different configurations of gaps.

58 This paper intends to propose an efficient method to compute the probability  
59 of failure of a tolerance analysis problem in the case of overconstrained mech-  
60 anisms. The technique is based on a transformation of the tolerance analysis  
61 problem formulation using the Lagrange duality property. This operation leads  
62 to an auxiliary problem which is free from the optimization step. The solution  
63 method includes a selective search algorithm so as to determine the dominant  
64 failure situations among the numerous possible ones. The probability of failure  
65 is eventually computed using the First Order Reliability Method (FORM) for  
66 systems [15, 16].

67 The paper is organized as follows: section 2 shows the current tolerance anal-  
68 ysis problem formulation whose probability is estimated thanks to the Monte  
69 Carlo simulation. Section 3 describes the mathematical transformation of the  
70 optimization problem into its Lagrange dual. Section 4 is devoted to the com-  
71 parison of the proposed method with the Monte Carlo simulation on different  
72 applications: first, the different transformation steps of the proposed formula-  
73 tion are detailed on a simple academic example. Then the method is applied to  
74 an overconstrained industrial application modeled in three dimensions.

## 75 **2. Tolerance analysis of overconstrained mechanisms**

### 76 *2.1. Problem formulation based on quantifiers*

The presence of gaps in overconstrained mechanisms makes the mechanical behavior difficult to model. Gaps are considered as free variables, but they are not free of constraints because interpenetration between two surfaces of two parts of the mechanism cannot be allowed. A set of  $N_C$  interface constraints are therefore defined to prevent surfaces from penetrating into each other. Let  $\mathbf{X} = \{X_1, \dots, X_n\}$  be the vector of random variables and  $\mathbf{g} = \{g_1, \dots, g_m\}$  the vector of gaps. Given a realization  $\mathbf{x}$  of the random vector  $\mathbf{X}$ , these constraints are inequations written as follows:

$$\{C_k(\mathbf{x}, \mathbf{g}) \leq 0\}_{k=1, \dots, N_C} \quad (1)$$

The functional condition equation of the mechanism is expressed as follows:

$$C_f(\mathbf{x}, \mathbf{g}) = y_{th} - Y \geq 0 \quad (2)$$

77 where  $Y = f(\mathbf{x}, \mathbf{g})$  is the response of the system (a parameter such as a gap or a  
 78 functional characteristic) modeled by a function  $f$  characterizing the influences  
 79 of the deviations and gaps on the mechanism behavior [17].

The universal quantifier “ $\forall$ ” (all) is used to translate the concept that the functional condition must be must respected in all configuration of the mechanism. The definition of the functionality of the mechanism is given by Qureshi *et al.* [1]: “for all admissible gap configurations of the mechanism, the geometrical behavior and the functional requirement are respected”. For any realization  $\mathbf{x}$  of the random vector describing the geometrical deviations, the mechanism is functional if the following holds:

$$C_f(\mathbf{x}, \mathbf{g}) \geq 0, \forall \mathbf{g} \in \mathbb{R}^m : C_k(\mathbf{x}, \mathbf{g}) \leq 0, \forall \mathbf{k} \in 1, \dots, N_C \quad (3)$$

The goal of tolerance analysis is to compute a probability of failure. From the previous definition of the functionality, the definition of the non functionality can be formulated as: “there exists at least one admissible configuration for which the functional condition is not respected”. That is why the worst configurations of gaps must be considered which is obtained when the minimum value of  $C_f(\mathbf{x}, \mathbf{g})$  is found whereas the interface constraints are satisfied. The expression of the probability of failure is given in Eq. (4).

$$P_f = \text{Prob}(R(\mathbf{X}) \leq 0) \quad (4)$$

80 where  $R(\mathbf{x}) = \min_{\mathbf{g} \in \mathbb{R}^m} C_f(\mathbf{x}, \mathbf{g})$  such that  $C_k(\mathbf{x}, \mathbf{g}) \leq 0, \forall \mathbf{k} \in 1, \dots, N_C$ .

## 81 2.2. First approach combining Monte Carlo simulation and optimization

A straight forward approach to estimate the probability of failure is based on the Monte Carlo simulation combined with an optimization algorithm. Let us introduce the indicator function  $\mathbb{1}_{\min C_f \leq 0}$  being equal to one if  $\min_{\mathbf{g}} C_f(\mathbf{x}, \mathbf{g}) \leq$

0 and zero otherwise. The probability of failure estimation is then expressed as follows:

$$\hat{P}_{fmc} = \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{\min C_f \leq 0}(\mathbf{x}^{(i)}) \quad (5)$$

where  $\{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(N)}\}$  is a set of samples from the random vector  $\mathbf{X}$ . This estimation technique, combined with the optimization scheme to evaluate  $\mathbb{1}_{\min C_f \leq 0}(\mathbf{x}^{(i)})$  for each  $(i)$ , is easy to implement and able to provide an accurate estimate of the probability provided that the coefficient of variation of the estimator, given Eq. (6), is small enough ( $<10\%$ ).

$$C.O.V_{\hat{P}_{fmc}} = \sqrt{\frac{1 - \hat{P}_{fmc}}{N \hat{P}_{fmc}}} \quad (6)$$

82 Monte Carlo simulation is the reference method, which can cope with a piecewise  
 83 continuous limit-state surface. For simple mechanisms, the optimization step  
 84 is costless although repeated  $N$  times. However, for highly overconstrained  
 85 systems, the optimization step makes the Monte Carlo simulation too much  
 86 time consuming, especially for small target probabilities. A new method is  
 87 therefore required in order to analyze complex systems faster.

### 88 **3. System reliability formulation based on Linear Programming**

89 As previously stated in the introduction, the full failure domain turns out  
 90 to be a system combinations of several failure domains relative to the different  
 91 configurations of gaps of the mechanism. Indeed, due to gaps and following the  
 92 realization of the random variables, several worst configurations of gaps may  
 93 exist in a mechanism when minimizing  $C_f(\mathbf{x}, \mathbf{g})$  [18]. In practice, some config-  
 94 urations often occur but some others may never happen. If several dominant  
 95 configurations are identified, then several discontinuities appears in the limit-  
 96 state surface. It represents a change of position of the mechanism parts from  
 97 a configuration to another. A worst configuration leading to a non functional  
 98 domain is called a failure situation. In [18], Beaucaire *et al.* propose a system  
 99 reliability formulation which can be obtained by solving the KKT optimality



100 conditions of the optimization problem given in Eq. (4). However this formu-  
 101 lation leads to define a probability formulation as the union of intersection of  
 102 events and cannot be used for complex mechanism.

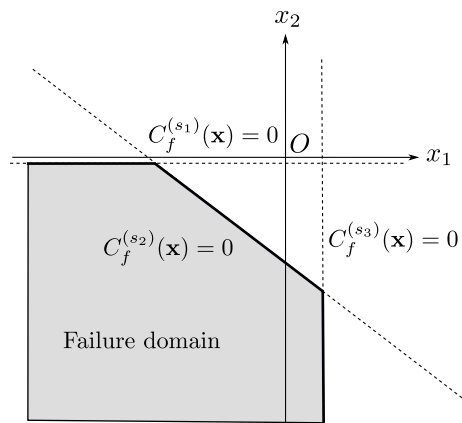


Figure 3: Limit-state surface for a mechanism with two random variables and three failure situations.

103 Figure 3 shows a limit-state surface for a mechanism with two random vari-  
 104 ables and three failure situations. It appears clearly that the probability of  
 105 failure matches the intersection of all failure domains of each situation. The  
 106 goal of this paper is to demonstrate this assumption from the current tolerance  
 107 analysis formulation and to propose a procedure to compute the probability of  
 108 failure  $P_f$  based on a system formulation. Subsection 3.1 shows the proposed  
 109 formulation of the tolerance analysis problem based on the Lagrange dual form  
 110 of the optimization problem. Subsection 3.2 describes the FORM method for  
 111 systems. Subsection 3.3 proposes a method in order to reduce the number of  
 112 intersection events by selecting only dominant failure situations.

### 113 3.1. Transformation of the optimization problem into the Lagrange dual form

The tolerance analysis problem formulation is defined as a linear function of gaps [1]. Indeed the functional condition is always defined as a linear combinations of several gap components. Interface constraints are most of the time also written as a linear functions of gaps [1]. However, cylinder type joints lead

to define quadratic interface constraints. In this case, constraints have to be linearized. This linearization of the behavior model has an impact on the probability of failure but this is not the purpose of the present paper. Given these properties, the optimization problem belongs therefore to the linear programming category. In addition, the objective function  $C_f$  is assumed differentiable. Given a realization  $\mathbf{x} = \{x_1, \dots, x_n\}$  of the random vector  $\mathbf{X}$ , let us define the Lagrange function:

$$L(\mathbf{x}, \mathbf{g}, \boldsymbol{\lambda}) = C_f(\mathbf{x}, \mathbf{g}) + \sum_{k=1}^{N_C} \lambda_k C_k(\mathbf{x}, \mathbf{g}) \quad (7)$$

114 where  $\boldsymbol{\lambda}$  are the  $N_C$  Lagrange multipliers [19].

115 **Proposition 3.1.** *In the present case,  $\mathbf{g}^*$  is an optimal solution of the opti-*  
 116 *mization problem if and only if there exists  $\boldsymbol{\lambda}^* \geq 0$  such that  $(\mathbf{g}^*, \boldsymbol{\lambda}^*)$  is a saddle*  
 117 *point of the Lagrange function in Eq. (7). In particular, the optimization prob-*  
 118 *lem has a solution if and only if  $L$  has saddle points, and if it does, then its*  
 119 *Lagrange dual problem has a solution whose optimal values are equal.*

120 Let us consider a Linear Programming problem, characterizing a tolerance anal-  
 121 ysis problem, which is written in a matrix form as follows:

$$\begin{aligned} \min_{\mathbf{g}} \quad & C_f(\mathbf{x}, \mathbf{g}) = \mathbf{a}^T \mathbf{x} + \mathbf{b}^T \mathbf{g} + c_0 \\ \text{subject to} \quad & C_k(\mathbf{x}, \mathbf{g}) = \mathbf{d}_k^T \mathbf{x} - \mathbf{e}_k^T \mathbf{g} + c_k \leq 0 \end{aligned} \quad (8)$$

122 for  $k = 1, \dots, N_C$  and for  $\mathbf{g} \in \mathbb{R}^m$ . The Lagrange function is given by:

$$\begin{aligned} L(\mathbf{x}, \mathbf{g}, \boldsymbol{\lambda}) &= C_f(\mathbf{x}, \mathbf{g}) + \sum_{k=1}^{N_C} \lambda_k C_k(\mathbf{x}, \mathbf{g}) \\ &= \left[ \mathbf{b} - \sum_{k=1}^{N_C} \lambda_k \mathbf{e}_k \right]^T \mathbf{g} + \sum_{k=1}^{N_C} \lambda_k (\mathbf{d}_k^T \mathbf{x} + c_k) + \mathbf{a}^T \mathbf{x} + c_0 \end{aligned} \quad (9)$$

123 Finally, the Lagrange dual problem reads:

$$\begin{aligned} \max_{\boldsymbol{\lambda}} \quad & C_{f,dual}(\mathbf{x}, \boldsymbol{\lambda}) = \sum_{k=1}^{N_C} \lambda_k (\mathbf{d}_k^T \mathbf{x} + c_k) + \mathbf{a}^T \mathbf{x} + c_0 \\ \text{subject to} \quad & \sum_{k=1}^{N_C} \lambda_k \mathbf{e}_k = \mathbf{b}; \boldsymbol{\lambda} \geq 0 \end{aligned} \quad (10)$$

124 The expression of the dual form allows the objective function to be no more  
125 a function of gaps. In addition, according to the proposition 3.1, if an opti-  
126 mal solution exists, then both optimization problems in Eqs. (8) and (10) have  
127 the same solution. The optimization constraints are now  $m$  equality equations  
128 to be satisfied and the Lagrange multipliers must be positive. The number of  
129 interface constraints  $N_C$  being greater than the dimension  $m$ , the dual opti-  
130 mization problem is under constraints: there are more Lagrange multipliers  $\lambda$   
131 than equality constraints. In order to find the admissible Lagrange multipliers  
132 satisfying the constraints, all possible combinations must be tested. However,  
some combinations are impossible. Table 1 lists several impossible situations.

Table 1: List of impossible combinations of Lagrange multiplier values.

Impossible situation	Physical interpretation
All $\lambda$ equal to 0 because $\mathbf{b} \neq 0$ .	This is equivalent to consider that there is no contact between the mechanism parts because no interface constraints can be equal to zero.
All $\lambda$ are different from 0 because $N_C \gg m$ .	This is equivalent to consider that there are contacts in every possible points of the mechanism simultaneously because all interface constraints must be equal to zero.

133

In fact, the number of non zero Lagrange multipliers  $\lambda_{s_k} = \{\lambda_{s_k^{(1)}}, \dots, \lambda_{s_k^{(m)}}\}$  must be equal to the number of gaps to be found, where  $s_k$  contains the indices of the Lagrange multipliers different from zero of the  $k^{th}$  situation. So, the number of possible situations is equal to  $N_s = \binom{N_C}{m}$ , but all situations may not be admissible regarding the optimization constraints, given in Eq. (10). Let  $N_{a,s}$  be the number of the admissible situations of Lagrange multipliers satisfying the given constraints and  $C_{f,dual}^{(s_k)}(\mathbf{x}) = C_{f,dual}(\mathbf{x}, \lambda_{s_k})$  be the expression of the dual functional condition relative to the combination  $s_k$ . There are then  $N_{a,s}$

admissible solutions to the maximization problem, in Eq. (10), which can now be written as follows:

$$C_{f,dual}^*(\mathbf{x}) = \max_{i=\{1,\dots,N_{as}\}} \left( \left\{ C_{f,dual}^{(s_i)}(\mathbf{x}) \right\} \right) \quad (11)$$

In addition, according to the proposition 3.1, the optimal values of the primal and dual optimization problem are equal, *i.e.* both expressions of the functional condition are equal:

$$C_f^*(\mathbf{x}) = C_{f,dual}^*(\mathbf{x}) = \max_{i=\{1,\dots,N_{as}\}} \left( \left\{ C_f^{(s_i)}(\mathbf{x}) \right\} \right) \quad (12)$$

134 Finally, the probability of failure is simply expressed as follows:

$$P_f = \text{Prob} \left( \max_i \left( \left\{ C_f^{(s_i)}(\mathbf{X}) \right\} \right) \leq 0 \right) \quad (13)$$

$$= \text{Prob} \left( \bigcap_{i=1}^{N_{as}} C_f^{(s_i)}(\mathbf{X}) \leq 0 \right) \quad (14)$$

$$= \text{Prob} \left( \left\{ C_f^{(s_1)}(\mathbf{X}) \leq 0 \right\} \cap \dots \cap \left\{ C_f^{(s_{N_{as}})}(\mathbf{X}) \leq 0 \right\} \right) \quad (15)$$

135 The above transformation procedure is described in detail in a simple example  
 136 in Subsection 4.1. This new formulation of the tolerance analysis problem as a  
 137 simple intersection of events allows the efficient FORM method for systems to  
 138 be applied to compute the probability of failure. This method is explained in  
 139 the following Subsection.

### 140 3.2. The FORM method for systems

The goal is to compute the probability from the multivariate Gaussian probability density function  $\phi_n$ , with the hypothesis of a first order approximation on each situation [15, 16]. Given  $N_{as}$  admissible failure situations, the parallel system failure probability, see Figure 4, is expressed as follows:

$$P_f = \text{Prob} \left( \bigcap_{k=1}^{N_{as}} C_f^{(s_k)}(\mathbf{X}) \leq 0 \right) \quad (16)$$

The reliability indices  $\beta_{s_k}$  and direction cosines  $\alpha_{s_k}$ , for  $k = 1, \dots, N_{as}$ , of each situation can be computed using the Hasofer-Lind-Rackwitz-Fiessler algorithm

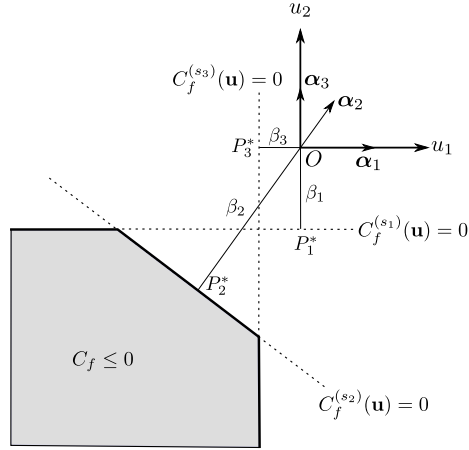


Figure 4: The FORM method for systems in the standard space with three failure domains.

[20] or the improved version [21]. In addition the matrix  $[\rho]$  of the limit-state correlation coefficients is gathered from the direction cosines:

$$\rho_{s_k s_j} = \boldsymbol{\alpha}_{s_k} \cdot \boldsymbol{\alpha}_{s_j} \quad (17)$$

141 The expression of the failure probability is given in Eq. (18).

$$\begin{aligned} P_f &= \text{Prob} \left( \bigcap_{k=1}^{N_{a.s}} C_f^{(s_k)}(\mathbf{X}) \leq 0 \right) \\ &= \Phi_{N_{a.s}}(-\boldsymbol{\beta}; [\rho]) \end{aligned} \quad (18)$$

142 where  $\Phi_{N_{a.s}}$  is the multivariate Gaussian cumulative distribution function. The  
 143 numerical value of this function can be evaluated using the method proposed  
 144 by Genz [22]. In addition, a confidence interval on the result is provided with  
 145 no extra numerical cost.

### 146 3.3. Reduction of the number of intersections

The admissible failure situations may be found by computing the Lagrange multipliers in all possible cases. This operation can be achieved if a small number of possibilities exist, *e.g.*  $N_s \leq 10$ . However for complex systems, the number of cases may exceed several thousands making this operation impossible. In addition, some admissible situations may have no influence on the failure

probability value. In both cases, finding and considering the  $N_{as}$  admissible situations is not conceivable. A search algorithm is therefore required in order to select only a number  $N_{ds}$  of dominant failure situations. The probability of failure is now expressed as follows:

$$P_f = \Phi_{N_{ds}}(-\beta; [\rho]) \quad (19)$$

147 Kim *et al.* [23] propose a simulation-based selective search algorithm which  
 148 uses a genetic algorithm. It is specially developed to find dominant failure modes  
 149 of a structure among a large number of possible modes. In the present paper,  
 150 the failure modes correspond to the failure situations. Considering the standard  
 151 space of the random variables, the dominant failure situations are located close  
 152 to the origin. Kim *et al.* propose an outward search in the standard space, from  
 153 points on a hypersphere near the origin to the points with a larger distance.  
 154 For a given hypersphere radius, several search directions in the standard space  
 155 are pseudo randomly defined. Points leading to a failure situation are saved  
 156 as elite chromosomes and an evolutionary operation is performed in order to  
 157 find additional failure situations in the same area (crossover) or in the opposite  
 158 direction (mutation). The evolutionary operation is repeated several times until  
 159 no new failure situations are found. The hypersphere radius is then increased in  
 160 order to search further away in the standard space. Once the hypersphere radius  
 161 is great enough or sufficient failure situations have been found, the algorithm  
 162 stops. The full algorithm is well described in [23].

#### 163 4. Applications

164 A simple academic example is proposed as a first illustration in Subsection  
 165 4.1. In particular, the proposed formulation with the transformation steps into  
 166 the Lagrange dual form is detailed. Subsection 4.2 shows the application of the  
 167 method to an industrial application. Results are compared with the reference re-  
 168 sults obtained with the Monte Carlo simulation combined with the optimization  
 169 scheme.

170 4.1. Illustration on a simple academic example

The optimization problem characterizing a pseudo behavior model in its primal form is defined in Eq. (20). The functionality is ensured when  $C_f \geq 0$ . The goal is to compute the probability  $P_f = \text{Prob}(\min_{\mathbf{g}} C_f(X_1, X_2, \mathbf{g}) \leq 0)$  where the random variables  $X_1$  and  $X_2$  follow a normal distribution  $\mathcal{N}(\boldsymbol{\mu}_X, \boldsymbol{\sigma}_X)$  whose parameters are given in Table 2. Three different set of parameters are used in order to change the order of magnitude of the probability failure.

$$\begin{aligned}
\min \quad & C_f(\mathbf{x}, \mathbf{g}) = x_1 + x_2 + 1 + g_1 + 2g_2 \\
\text{subject to} \quad & C_1(\mathbf{x}, \mathbf{g}) = x_1 - 1 - g_1 \leq 0 \\
& C_2(\mathbf{x}, \mathbf{g}) = x_2 + 2 - g_2 \leq 0 \\
& C_3(\mathbf{x}, \mathbf{g}) = x_1 - x_2 - 2g_1 \leq 0 \\
& C_4(\mathbf{x}, \mathbf{g}) = 2x_1 + x_2 + g_1 - 2g_2 \leq 0
\end{aligned} \tag{20}$$

Table 2: The three set of parameters of the normal random variables for the academic example.

Set of parameters	$\mu_X$		$\sigma_X$	
	All	1	2	3
$X_1$	0	1	0.5	0.4
$X_2$	0	1	0.5	0.4

From the definition of the Lagrange dual problem given in Eq. (10), the dual form of this optimization problem is written as follows:

$$\begin{aligned}
\max_{\boldsymbol{\lambda}} \quad & C_{f,dual}(\mathbf{x}, \boldsymbol{\lambda}) = \lambda_1(x_1 - 1) + \lambda_2(x_2 + 2) + \lambda_3(x_1 - x_2) \\
& \quad \quad \quad + \lambda_4(2x_1 + x_2) + x_1 + x_2 + 1 \\
\text{subject to} \quad & \lambda_1 + 2\lambda_3 - \lambda_4 = 1 \\
& \lambda_2 + 2\lambda_4 = 2 \\
& \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0
\end{aligned} \tag{21}$$

171 The number of Lagrange multipliers  $\boldsymbol{\lambda}$  is greater than the number of equality  
172 constraints so the different possibilities where two  $\lambda$  are different from zero (and  
173 the others are set to zero) must be tested, *i.e.* a number of possible situations

174  $N_s = \binom{4}{2} = 6$ . Lagrange multipliers values  $\lambda$  are computed solving the linear  
175 system of equality constraints. Table 3 lists these possible situations and shows  
176 if the result is admissible or not.

Table 3: List of possible combinations to compute Lagrange multiplier values.

Situation hypothesis	Result	Expression of $C_f$
$s_1$ : $\lambda_{3,4} \neq 0$ $\lambda_{1,2} = 0$	$\lambda_3 = 1$ $\lambda_4 = 1$	$C_f^{s_1} = 4x_1 + x_2 + 1$
$s_2$ : $\lambda_{2,4} \neq 0$ $\lambda_{1,3} = 0$	$\lambda_4 = -1 < 0$ $\lambda_2 = 4$	It is not an optimum
$s_3$ : $\lambda_{2,3} \neq 0$ $\lambda_{1,4} = 0$	$\lambda_2 = 2$ $\lambda_3 = 1/2$	$C_f^{s_3} = \frac{3x_1 + 5x_2}{2} + 5$
$s_4$ : $\lambda_{1,4} \neq 0$ $\lambda_{2,3} = 0$	$\lambda_1 = 2$ $\lambda_4 = 1$	$C_f^{s_4} = 5x_1 + 2x_2 - 1$
$s_5$ : $\lambda_{1,3} \neq 0$ $\lambda_{2,4} = 0$	Constraint of Eq. (21) $\Rightarrow 2 = 0$	Impossible
$s_6$ : $\lambda_{1,2} \neq 0$ $\lambda_{3,4} = 0$	$\lambda_1 = 1$ $\lambda_2 = 2$	$C_f^{s_6} = 2x_1 + 3x_2 + 4$

177 The admissible failure situations are found in this case by computing the  
178 Lagrange multipliers in order to describe in detail why some configurations are  
179 not admissible. The selective search algorithm, see Subsection 3.3, can also  
180 be used. From Table 3, the number of admissible situations is  $N_{as} = 4$ . The  
181 optimization problem is now written as follows:

$$\begin{aligned}
C_f^*(\mathbf{x}) &= \max \left( C_f^{s_1}(\mathbf{x}), C_f^{s_3}(\mathbf{x}), C_f^{s_4}(\mathbf{x}), C_f^{s_6}(\mathbf{x}) \right) \\
&= \max \left( 4x_1 + x_2 + 1, \frac{3x_1 + 5x_2}{2} + 5, 5x_1 + 2x_2 - 1, 2x_1 + 3x_2 + 4 \right)
\end{aligned} \tag{22}$$

The associated probability of failure is only a function of random variables  $X_1$



and  $X_2$ , as seen from the following expression:

$$P_f = \text{Prob} \left( \begin{array}{l} \{4X_1 + X_2 + 1 \leq 0\} \cap \left\{ \frac{3X_1 + 5X_2}{2} + 5 \leq 0 \right\} \\ \cap \{5X_1 + 2X_2 - 1 \leq 0\} \cap \{2X_1 + 3X_2 + 4 \leq 0\} \end{array} \right) \quad (23)$$

The FORM method is applied on each failure situation of the previous formulation. In this case of linear functions with normal random variables, the FORM method provides exact results of each reliability index. Results obtained for the first set of parameters are shown in Table 4. The correlation matrix associated with this problem is given in Eq. (24).

Table 4: Reliability indices obtained for the first set of parameters for the academic example.

Failure situation	$s_1$	$s_3$	$s_4$	$s_6$
$\beta$	0.24	1.71	-0.19	1.12

$$[\boldsymbol{\rho}] = \begin{bmatrix} 1 & 0.707 & 0.998 & 0.796 \\ 0.707 & 1 & 0.739 & 0.991 \\ 0.998 & 0.739 & 1 & 0.824 \\ 0.796 & 0.991 & 0.824 & 1 \end{bmatrix} \quad (24)$$

Given these values, the probability of failure can be computed using the multivariate Gaussian cumulative distribution function in four dimensions:

$$P_f = \Phi_4(-\{\beta_{s_1}, \beta_{s_3}, \beta_{s_4}, \beta_{s_6}\}; [\boldsymbol{\rho}]) \quad (25)$$

182 The comparison of the results obtained with the Monte Carlo simulation  
183 (for a C.O.V  $\approx 5\%$ ) and the FORM method for systems are shown in Table 5  
184 for the three set of parameters. The probability of failure is expressed in parts  
185 per million (ppm). The smallest the probability, the longer the Monte Carlo  
186 simulation to be accurate enough. In contrast, using the proposed formulation  
187 and solution method, the computing time is small and stable. Results provided  
188 by the FORM method for systems are consistent with those obtained by the  
189 Monte Carlo simulation. In addition, the selective search algorithm appears  
190 useful because only two failure situations are selected as dominant situations,

191 which allows again to save time. Indeed, only the failure situations  $s_1$  and  $s_3$   
 192 are required to compute the probability of failure. As shown in Figure 5, the  
 193 failure domain is the intersection of two failure domains corresponding with the  
 194 situations  $s_1$  and  $s_3$ .

Table 5: Comparison between Monte Carlo simulation and FORM method for systems results for the academic example.

	Monte Carlo	FORM system
	42300	41214
$P_d$ ( $\times 10^{-6}$ )	307	301
	9.06	9.03
	8050	
95% C.I. ( $\times 10^{-6}$ )	57	
	1.7	
	4 s	0.7 s
Computing time	11 min	0.7 s
	6.2 h	0.6 s

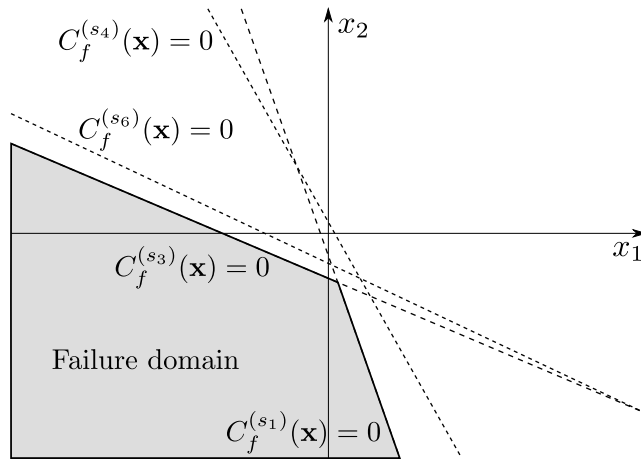


Figure 5: Failure domain of the academic example. The failure domain is the intersection of two failure domains of situations  $s_1$  and  $s_3$ .

195 *4.2. Industrial application*

196 The application is based on a gear pump, see Figure 6, which has two parts  
 197 positioned with two pins. The positioning of these two parts has an influence on  
 198 the angle of both gear axes. The functionality of the pump can be reduced if the  
 199 assembly precision of the parts is insufficient. Based on this pump, a simplified  
 200 overconstrained mechanism is studied. Figure 7 shows the mechanism with  
 201 amplified gaps between parts. The functional condition concerns the deviation  
 202 of the point  $G$  of part (1) with respect to part (2). This point  $G$  can be seen as  
 a functional point that is representative of one axis of the gear pump.

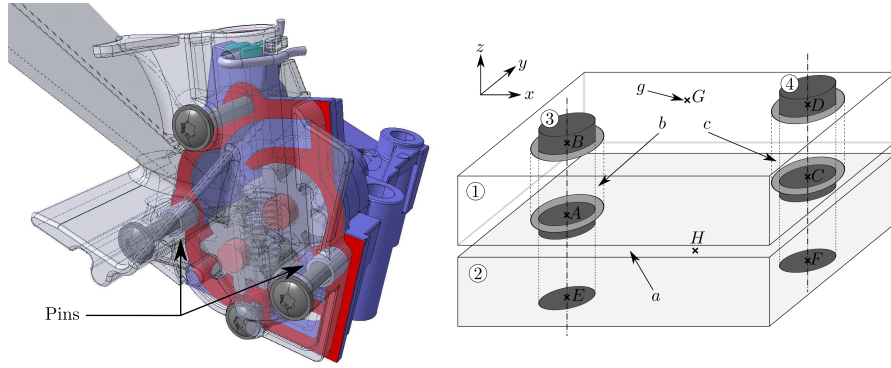


Figure 6: Full pump view.

Figure 7: Simplified mechanism in 3D.

203

204 Characteristics of the mathematical behavior model are listed below:

- 205 • 38 random variables following a Gaussian distribution  $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}_X, \boldsymbol{\sigma}_X)$ .
- 206 Values of all parameters are given in Appendix A.
- 207 • 3 gap variables  $\mathbf{g}$  which are the optimizations parameters.
- 208 • 4 quadratic interface constraints which give  $N_C = 160$  interface constraints
- 209 after applying a linearization procedure.
- 210 •  $N_s = \binom{160}{3} = 669920$  possible situations.

211 In order to show the efficiency of the proposed formulation, different orders  
 212 of magnitude of the probability of failure are intended to be reached. Probabilities  
 213 of failure are expressed in parts per million (ppm). Table 6 shows the

214 numerical results obtained for three levels of probability. The maximum number  
 215 of dominant failure situations,  $N_{ds}$ , found by the selective search algorithm is  
 216 set to 5, 100 and 200. A number of failure situations too small leads to a loss  
 217 of accuracy of the probability of failure whereas a too large number leads to a  
 218 waste of time in calculation because there is no need to increase the precision  
 219 on the result. In cases with a large  $N_{ds}$ , it can be seen that the FORM method  
 220 for systems allows to get accurate results that are identical to the reference re-  
 221 sults with the Monte Carlo simulation. The obtained values of the confidence  
 222 interval are small enough to ensure an accurate result. In addition, reaching  
 223 low probabilities with precision is also possible within an acceptable comput-  
 224 ing time. Results are obtained in less than two minutes whereas the Monte  
 225 Carlo simulation requires several hours. Indeed, low probabilities lead to define  
 226 a large Monte Carlo population so as to yield an accurate estimation of the  
 227 probability of failure, thus greatly increasing the computing time because of the  
 optimization step.

Table 6: Comparison between Monte Carlo simulation and FORM method for systems results for the industrial application.

	Monte Carlo	FORM system		
$N_{ds}$		5	100	200
	17933	23889	18239	18289
$P_d$ ( $\times 10^{-6}$ )	155	282	156	154
	9.4	16	9.3	9.1
	3065			
95% C.I.	28			
	1.7			
	3 min	22 s	66 s	2.15 min
Computing time	5.1 h	23 s	67 s	2.26 min
	3.5 d	26 s	73 s	2.35 min

228

## 229 5. Conclusion

230       Considering that a mechanism behavior of a product is disturbed by admis-  
231 sible geometrical deviations and gaps between surfaces, it is natural to dispose  
232 of a tool able to evaluate the quality level of a designed mechanism. The toler-  
233 ance analysis method takes into account the specified design tolerances in order  
234 to determine whether they ensure a mechanism to be assembled and functional.  
235 Evaluating the quality level turns out to be an estimation of a probability of  
236 failure which must be done as fast and accurate as possible. The standard  
237 method, which combines Monte Carlo simulation and an optimization scheme,  
238 is not able to provide accurate result and small values of ppm in a reasonable  
239 computing time for complex overconstrained mechanism. This paper proposes  
240 a new procedure to deal with a functional tolerance analysis.

241       The proposed procedure is based on a system formulation of the problem  
242 so that a system reliability method can be used. This formulation is obtained  
243 using the Lagrange dual form of the original optimization problem. The system  
244 formulation of the problem is defined as a simple intersection of events. A  
245 simulation-based selective search algorithm is used in order to find only the  
246 dominant events. The probability of failure is then quickly estimated using the  
247 FORM method for systems.

248       An academic example is used to detail all the transformation procedure.  
249 Then the proposed method is applied on an industrial application modeled in  
250 three dimensions. Results show that the FORM method for systems provides  
251 accurate result much faster than with the Monte Carlo simulation. In addition,  
252 the procedure is able to handle problems where low probabilities are intended  
253 to be reached. The application remains simple enough to be able to perform a  
254 Monte Carlo simulation so as to compare results. The major advantage of the  
255 proposed method is for highly complex systems, a functional condition com-  
256 puted using a finite element modeling for instance, in this case Monte Carlo  
257 simulation cannot be applied anymore. The proposed procedure is developed to  
258 solve tolerance analysis problems, though it is conceivable that other research

259 fields may have problems written as the original formulation. In this case, the  
 260 transformation of the formulation into a system formulation may also be applied.

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264 **Appendix A. Parameters of the normal random variables of the in-**  
 265 **dustrial application**

Set of parameters	$\mu_X$		$\sigma_X$	
	All	1	2	3
$y_{th}$		0.25	0.28	0.29
$X_1$	20	0.06	0.03	0.02
$X_2$	19.8	0.06	0.03	0.02
$X_3$	20	0.06	0.03	0.02
$X_4$	19.8	0.06	0.03	0.02
$X_5$	0	0.01	0.01	0.01
$X_6$	0	0.01	0.01	0.01
$X_7$	0	0.01	0.01	0.01
$X_8$	0	0.01	0.01	0.01
$X_9$	0	0.01	0.01	0.01
$X_{10}$	0	0.01	0.01	0.01
$X_{11}$	0	0.01	0.01	0.01
$X_{12}$	0	0.01	0.01	0.01
$X_{13}$	0	0.01	0.01	0.01
$X_{14}$	0	0.01	0.01	0.01
$X_{15}$	0	0.01	0.01	0.01
$X_{16}$	0	0.01	0.01	0.01
$X_{17}$	0	0.01	0.01	0.01

Set of parameters	$\mu_X$		$\sigma_X$	
	All	1	2	3
$X_{18}$	0	0.01	0.01	0.01
$X_{19}$	0	0.01	0.01	0.01
$X_{20}$	0	0.01	0.01	0.01
$X_{21}$	0	0.01	0.01	0.01
$X_{22}$	0	0.01	0.01	0.01
$X_{23}$	0	0.01	0.01	0.01
$X_{24}$	0	0.01	0.01	0.01
$X_{25}$	0	0.01	0.01	0.01
$X_{26}$	0	0.01	0.01	0.01
$X_{27}$	0	0.001	0.001	0.001
$X_{28}$	0	0.001	0.001	0.001
$X_{29}$	0	0.001	0.001	0.001
$X_{30}$	0	0.01	0.01	0.01
$X_{31}$	0	0.01	0.01	0.01
$X_{32}$	0	0.01	0.01	0.01
$X_{33}$	0	0.001	0.001	0.001
$X_{34}$	0	0.001	0.001	0.001
$X_{35}$	0	0.001	0.001	0.001
$X_{36}$	0	0.01	0.01	0.01
$X_{37}$	0	0.01	0.01	0.01
$X_{38}$	0	0.01	0.01	0.01

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