A New Fast Track to Nonlinear Modal Analysis of Power System Using Normal Form

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***Abstract*—The inclusion of higher-order terms in small-signal (modal) analysis augments the information provided by linear analysis and enables better dynamic characteristic studies on the power system. This can be done by applying Normal Form theory to simplify the higher order terms. However, it requires the preliminary expansion of the nonlinear system on the normal mode basis, which is impracticable with standard methods when considering large scale systems. In this paper, we present an efficient numerical method for accelerating those computations, by avoiding the usual Taylor expansion. Our computations are based on prescribing the linear eigenvectors as unknown field in the initial nonlinear system, which leads to solving linear-only equations to obtain the coefficients of the nonlinear modal model. In this way, actual Taylor expansion and associated higher order Hessian matrices are avoided, making the computation of the nonlinear model up to third order and nonlinear modal analysis fast and achievable in a convenient computational time. The proposed method is demonstrated on a single-machine-infinite-bus (SMIB) system and applied to IEEE 3-Machine, IEEE 16-Machine and IEEE 50-Machine systems.**

***Index Terms*—Modal analysis, Normal Form coefficients, Power system dynamics, Reduced computation, Selective nonlinear analysis.**

I. Introduction

THE emergence of new power electronic-based (PE) devices onto the power scene and the increasing number of distributed generation power systems in electrical grids contribute to changing the characteristic behaviour of the grid. This outburst seems to be the fastest growing trend in power system with proliferation of DC/AC inverters, switch mode power supplies, high-voltage DC (HVDC) links, distributed renewable energy systems and many more [1]–[3]. The consequence is a complex nonlinear behaviour and failure of conventional small-signal (modal) analysis (SSA) tools [4]–[6]. The inclusion of higher-order terms, which leads to Nonlinear Modal Analysis (NMA) augments the information provided by SSA and enables better dynamic characteristic studies of the power systems.

The two currently used techniques for NMA are Normal Form (NF) and Modal Series (MS) methods [6]–[18]. Although, there seems to be an argument regarding which method is better in literature, NF is widely used [19], [20].

Normal Form method has remained a veritable tool in power system studies and has been extensively used. In the last two decades, NF results had revealed that the interactions among modes have strong effects in the control designs and transient stability of power system [12], [21]. In testimony of this, better nonlinear control designs using NF are being developed [17], [22]–[24]. The authors in [25] showed that power system stability sub-modes interact with each other, especially when the power system presents strong nonlinear characteristics. Stability sub-modes were defined as the modes which participate more in voltage and angle state variables. Various stability indices using NF have been proposed in [5], [16], [26]. With NF, the inter-area separation and grouping of generators following disturbance can be better predicted as shown in [27], [28]. The effectiveness of NF for studying grids with high penetration of renewable energies has currently been reported [3]. There are indeed so many applications of Normal Form in power system.

However, this precious technique has been challenged by its computational complexity arising from the inclusion of higher order terms in the Taylor series expansion. As a result, most studies are based on inclusion of 2nd order terms with few works on inclusion of third order terms. Current researches emphasize the deficits of 2nd order NF in the light of non-linearly growing power systems [5], [8], [18]. Moreover, 2nd order NF cannot be used for stability studies [6]. In search of alternative, the use of Koopman Mode Decomposition (KMD) based on measured data is being investigated [29]. However, a comparison of both approaches showed same computation complexity [30]. A Very recent work reveals that the efficiency of KMD largely depends on the choice of the observable state variables and one is never sure the exact states to observe to get good results [29]. This review motivates the need to expand Normal Form analysis to adapt to more complex system representations.

The major difficulty in NF is encountered in: 1) Building the approximate model and computing its numerous nonlinear coefficients, and 2) Finding the initial condition in Normal Form space. These coefficients increase exponentially with system

size. In order to exploit in details, the benefits of Normal Form, several types of power system models such as grids including virtual synchronous machines (VSM), HVDC links, wind farms, and PVs have to be investigated. This means that tools developed for NF have to be easily adapted to a change of model in order to be applicable to different types of grid. The conventional method for building Normal Form approximate models is to expand the system equations by Taylor series up to desired order. This gives rise to higher order derivatives and Hessian matrices evaluated at the operating point. Generally, four approaches may be used to build the Hessian matrices and then compute NF coefficients: 1) Symbolic tools can be used to perform the higher order Hessian derivative as done in [31]; 2) The expressions for the Hessian derivatives may be pre-defined and then stored as a library in advance as done in [32] for 2nd order NF; 3) Perhaps, numerical differentiation technique is achievable; 4) Automatic differentiation (AD) method also exists [33]–[37]. Automatic differentiation is a method for efficiently augmenting computer programs with statements for the computation of derivatives; motivated by the fact that every computation is made up of elementary mathematical operations like $\sin, \cos$, multiplications, and so on [33]. Considering the various combinations of these elementary operations, AD exploits the structure of the chain rule to evaluate differential operators of a function as the function itself is being evaluated as a computer program [35]. Symbolic computation minimizes error while dealing with higher order differentiation, but it is generally too slow and does not scale to large system. If the expressions for the Hessian derivatives are defined in advance, only substitution of the operating point is needed. As a result, this approach is expected to be faster once the expressions are defined. However, any change in the model overhauls the whole exercise. At 3rd order, this approach can be cumbersome. It can take days and it is easy to make mistakes after the painstaking work. In this aspect, this method is very conservative and therefore not good for different models. This is probably the reason why most NF applications to a relatively large system follow similar power system model (see [24], [30]). Numerical differentiation simplifies derivative problems, however it accumulates error. The higher the derivative, the more problematic is the round-off error. For 2nd and higher-order derivatives where possible, and often, for cross-derivatives, the error can become substantial. This is not desirable especially since the Taylor series is in itself an approximation. AD overcomes the problems of numerical and symbolic differentiation and has similar efficiency as hand derived derivatives but there are also some concerns. AD implementation requires high level skills almost reserved for the experts and AD software developers. Decomposing a computer program into the necessary component functions required in AD is far from simple. A computationally naive implementation of AD can result in outrageously slow code and excessive use of memory [34], [36]. There exists no standard set of problems cutting across the varieties of AD applications and the development of a package is usually driven by a specific class of problems [34]. AD implementation for higher-order derivatives is very complicated and far from the well established first-order implementation. A common method is to repeat AD of the first order till the desired order is obtained, an approach which Betancourt [35] described as inefficient.

For polynomial nonlinearity, such as assumed in NF, not all terms will be very important for some studies. This necessitates selective evaluation of certain terms in the polynomial approximation. With selective NF application in view, the above methods are not very convenient.

In this paper, we propose a fast method to obtain the needed nonlinear coefficient for NF model without Taylor expansion and the associated Hessian matrices computation. This method is simple and is based on prescribing the linear eigenvectors as unknown field in the initial nonlinear system, which leads to solving linear-only equations to obtain the coefficients of the nonlinear modal model. It was introduced in the mechanical engineering field in [38] and widely applied since, to compute the coefficients of nonlinear modal reduced order models of mechanical structures discretized by a finite-elements method [39], [40]. It is particularly attractive since a standard commercial finite-elements code can be used in a so called non-intrusive way. However, to the knowledge of the authors, this method has been applied only to systems of second order differential equations with real eigenvalues/vectors, standard for mechanical systems. On the contrary, power systems are modelled usually as first order systems with complex eigenvalues/vectors with non zero operating points. The originality of the present work is to extend this method to first order complex differential systems. The focus of this paper is on the development of a new methodology that avoids the use of a Taylor series expansion rather than on examining specific differential algebraic equations (DAEs) power system models.

II. Normal Form Technique

A. General Idea of NF Application to Power Systems

Normal Form is a mathematical technique that simplifies a set of nonlinear differential equations into a simplified one, that can be linear in some particular cases, by making sequential nonlinear coordinate transformations. The resulting equations are then in their simplest form (Normal Form) [41]–[45]. The basic steps of its application to power systems can be summarized below:

1) *Formulate the DAEs of the system*: The power flow solutions are obtained and the stable equilibrium point (SEP) for the post-fault system are determined.
2) *Taylor expansion:* The algebraic equations are substituted into the differential ones and the resulting differential-only equations are expanded by Taylor series up to desired order around the SEP. The Taylor expansion can also be done on structure preserving power system DAEs [9].
3) *Linear analysis:* The Jacobian of the system is used to extract the system eigenvalues and vectors.
4) *Modal expansion:* The system of step 2 is expanded onto the eigenvectors basis obtained at step 3 and a new dynamical system with Jordan linear part is obtained,

with nonlinear terms that couple the equations. The coefficients of the nonlinear terms have to be computed.

5) *Normal form transformation:* The nonlinear part is further simplified by applying the normal form technique, based on successive nonlinear change of variables.
6) *NF initial conditions and simulation:* For transient simulations, the initial condition of the NF system are determined, usually by combination of Newton-Raphson method and optimization techniques. The system is then simulated in NF coordinates.
7) *Inverse NF and modal transformation:* The original dynamics can be reconstructed by using the change of coordinates of steps 4 and 5).

For power system applications, step 7 is often only used as a verification of the method. Meanwhile, most analyses are based on the investigation of the information in steps 3 to 6. Step 5 finally boils down to simple division of the nonlinear coefficients (computed in step 4) by different combinations of the linear eigenvalues (computed in step 3). Hence the major computational burden is in step 2 to step 4 and step 6. The method promoted in the present work aims at significantly simplifying steps 2 to 4.

B. Normal Form Theory

Consider a first order modelling of a N-dimensional power system as

$$\dot{\mathbf{X}} + \mathbf{f}(\mathbf{X}) = \mathbf{P}_T, \tag{1}$$

where $\mathbf{X}$ and $\mathbf{P}_T$ are vectors of state and external input respectively. $\mathbf{f}(\mathbf{X})$ is nonlinear and its expression and degree of nonlinearity depend on the particular model adopted. It is assumed in (1) that the algebraic equations are already substituted in the differential ones and (1) is differential-only. The application of NF to power system structure preserving DAE models has been addressed before [9], but that is outside the scope of the present paper and would be considered in future work. For small perturbation, (1) can be expanded in Taylor series around a particular SEP $\mathbf{X}_0$. One can write $\mathbf{X}(t) = \mathbf{X}_0 + \mathbf{x}(t)$ and, neglecting terms of order higher than 3, we obtain, with Einstein notations [46]:

$$\dot{x}_i + A_{ij}x_j + F2_{ijk}x_jx_k + F3_{ijkl}x_jx_kx_l = P_i \tag{2}$$

where x_i is the i-th component of vector $\mathbf{x}$ and, for all $i, j, k, l = 1, 2, \dots N$:

$$A_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\mathbf{X}=\mathbf{X_0}}, \quad F2_{ijk} = \frac{1}{2}\left.\frac{\partial^2 f_i}{\partial x_j \partial x_k}\right|_{\mathbf{X}=\mathbf{X_0}}, \tag{3a}$$

$$F3_{ijkl} = \frac{1}{6}\left.\frac{\partial^3 f_i}{\partial x_j \partial x_k \partial x_l}\right|_{\mathbf{X}=\mathbf{X_0}}, \quad P_i = P_{Ti} - f_i(\mathbf{X}_0). \tag{3b}$$

A, **F2**, and **F3** are respectively the Jacobian of size $(N \times N)$, 2nd order Hessian of size $(N \times N \times N)$ and 3rd order Hessian of size $(N \times N \times N \times N)$ corresponding to 1st, 2nd, and 3rd order terms evaluated at SEP $\mathbf{X_0}$. Equation (2) can be expressed compactly as

$$\dot{\mathbf{x}} + \mathbf{A}\mathbf{x} + \boldsymbol{\beta}(\mathbf{x}) = \mathbf{P}, \tag{4}$$

where $\boldsymbol{\beta}(\mathbf{x})$ collects all second and third order terms.

Let us denote by $\mathbf{U}_i$, and $\mathbf{V}_i$ the i-th columns of right and left eigenvectors respectively, $\mathbf{U} = [u_{ij}]$, $\mathbf{V} = [v_{ij}]$, the corresponding matrices and $\boldsymbol{\Lambda} = \mathbf{V^T A U} = \mathrm{diag}(\lambda_p)$, the diagonal matrix of its eigenvalues, $p, i, j = 1, \dots N$. It is assumed that the matrix $\mathbf{A}$ is diagonalizable. Using the linear transformation:

$$\mathbf{x} = \mathbf{U}\mathbf{y} \tag{5}$$

in (4) and multiplying the result by the left eigenvectors yields:

$$\dot{\mathbf{y}} + \boldsymbol{\Lambda}\mathbf{y} + \mathbf{f}_{\mathrm{NL}}(y_1, y_2, \dots, y_N) = \mathbf{V^T P}, \tag{6}$$

or, for all $p = 1, \dots N$:

$$\begin{aligned} \dot{y}_p + \lambda_p y_p + \sum_{q=1}^{N}\sum_{r=1}^{N} C_{pqr}y_q y_r + \dots \\ \sum_{q=1}^{N}\sum_{r=1}^{N}\sum_{s=1}^{N} D_{pqrs}y_q y_r y_s = \mathbf{V}_p^T\mathbf{P}, \end{aligned} \tag{7}$$

where

$$\mathbf{f}_{\mathrm{NL}} = \mathbf{V^T}\boldsymbol{\beta}(\mathbf{U}\mathbf{y}) \tag{8}$$

and $D_{pqrs} = F3_{ijkl}v_{ip}u_{jq}u_{kr}u_{ls}$, $C_{pqr} = F2_{ijk}v_{ip}u_{jq}u_{kr}$ are the 2nd and 3rd order nonlinear coefficients in modal coordinate. In power system modal analysis, the external input is often assumed to be constant for small disturbance and therefore, the right hand side of (6) is usually assumed zero. However, as it will be seen later, the proposed method needs a non constant external input.

If constant external input is assumed, (7) can be written in a more compact form as:

$$\dot{\mathbf{y}} + \boldsymbol{\Lambda}\mathbf{y} + \mathbf{C}(\mathbf{y}) + \mathbf{D}(\mathbf{y}) = \mathbf{0}. \tag{9}$$

NF theory leads to simplifying the nonlinear terms in (9) by a nonlinear polynomial change of variables, written as [44]:

$$\mathbf{y} = \mathbf{z} + \mathbf{h2}(\mathbf{z}) + \mathbf{h3}(\mathbf{z}), \tag{10}$$

where $\mathbf{z}$ is the state variable in NF coordinate, $\mathbf{h2}$ and $\mathbf{h3}$ are respectively complex valued quadratic and cubic polynomials in $\mathbf{z}$ with $h2^j_{kl}$ and $h3^j_{pqr}$ coefficients determined such that (9) is simplified. Substituting (10) into (9) yields

$$\dot{\mathbf{z}} + \boldsymbol{\Lambda}\mathbf{z} + \hat{\mathbf{F}}(\mathbf{z}) = \mathbf{0}. \tag{11}$$

where

$$\begin{aligned} \hat{\mathbf{F}}(\mathbf{z}) = \mathbf{C}(\mathbf{z}) + \mathbf{D}(\mathbf{z}) + \boldsymbol{\Lambda}\mathbf{h_2}(\mathbf{z}) - \bar{\mathbf{D}}\mathbf{h_2}(\mathbf{z})\boldsymbol{\Lambda}\mathbf{z} \\ + \boldsymbol{\Lambda}\mathbf{h_3}(\mathbf{z}) - \bar{\mathbf{D}}\mathbf{h_3}(\mathbf{z})\boldsymbol{\Lambda}\mathbf{z} \end{aligned}$$

and $\bar{\mathbf{D}}$ is a derivative operator, with terms of order higher than three neglected. Removing nonlinear terms from (11) implies setting the third term on the left hand side of (11) to zero. Then $\mathbf{h_2}$ and $\mathbf{h_3}$ can be determined as [44]

$$h2_{pqr} = \frac{C_{pqr}}{\lambda_q + \lambda_r - \lambda_p}, h3_{pqrs} = \frac{D_{pqrs}}{\lambda_q + \lambda_r + \lambda_s - \lambda_p}. \tag{12}$$

It can be seen that if the denominators of the change of variables in (12) are close to 0 with non zero absolute value of the numerators, the value of the coefficients will be very large,

which would lead to a non consistent change of variables. The nullity of the denominators leads to particular relations between the complex eigenvalues called internal resonances. If no internal resonance occurs, (11) reduces to a fully linear system:

$$\dot{\mathbf{z}} = \mathbf{\Lambda z}. \tag{13}$$

In the other cases where internal resonances occur, all nonlinear terms cannot be eliminated from (11) and (13) becomes

$$\dot{\mathbf{z}} = \mathbf{\Lambda z} + \mathbf{g}(\mathbf{z}). \tag{14}$$

In the above equation, $\mathbf{g}(\mathbf{z})$ is a collection of all the so called resonant terms that cannot be eliminated. If 3rd order nonlinear terms are considered and there exist lightly damped modes, there are always inherent near resonances so that $\mathbf{g}(\mathbf{z})$ is never zero. Such near resonance exist because if $\lambda_i, \lambda_{2i-1}, \forall i \in M$ lightly damped modes are complex conjugates, $\lambda_i + \lambda_{2i-1} + \lambda_p - \lambda_p$ will always lead to very small divisor (near resonance) which translates to ill-conditioning. Technique for handling internal resonances is outside the scope of this paper and the interested reader can refer to [20]. At the end, system (14) has two advantages. Due to the normal transform, it is first much simpler than system (2) since it has much less nonlinear terms and second, it can be consistently truncated to a few modes, thanks to the concept of nonlinear modes and invariant manifolds (see [45], [47], [48] and references therein). System (14) can then be used for nonlinear modal analyses, time integration and many more, and can be reversed to its original coordinates $\mathbf{x}$ by applying successively, the change of variables (10) and (5).

C. Computational Burden

For better understanding of the main issue this paper aims at solving, we can compute the number of nonlinear coefficients required. Considering an N-dimensional dynamical system and the symmetries of the Hessian matrices in (3), the number of nonlinear coefficients (which is equivalent to number of computations) that need to be computed in $\boldsymbol{\beta}(\mathbf{x})$ (Eq. (4)) is:

$$N_c = N\left[\frac{(N+1)!}{2!(N-1)!} + \frac{(N+2)!}{3!(N-1)!}\right] = \frac{N^4}{6} + N^3 + \frac{5N^2}{6}. \tag{15}$$

where $N!$ is the factorial of integer N. Furthermore, the number of coefficients C_{pqr} and D_{pqrs} in the modal model (7) that have to be computed using (8) is again N_c. Equation (15) shows that the computational burden increases with the power of four of the number of degrees of freedom of the initial system. Then, the overall implication in terms of memory and computation time can be huge, even impracticable for large systems with large N. As an illustration, the evolution of the number of coefficients with respect to the linear basis size is shown in Fig.1. A small change in size leads to a huge computational burden.

The major advantage of the method proposed in this paper is that it enables one to compute directly, the nonlinear coefficients C_{pqr} and D_{pqrs} without requiring the preliminary computation of the Taylor expansion of Eq. (3) and the use of Eq. (8). Moreover, the final number of nonlinear coefficients

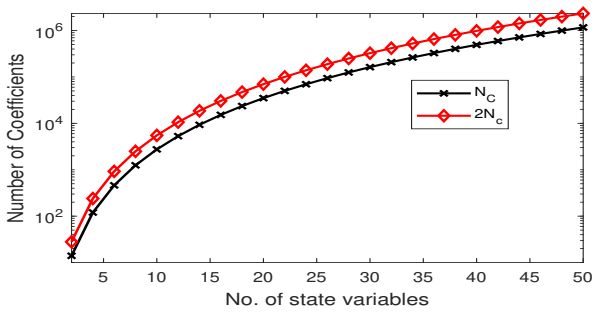

Fig. 1. 3rd Order Normal Form Computational Burden

in the modal model (7) is equal to N_c since it is naturally written in upper triangular form (The sums $\sum_{k,l=1}^{N}, \sum_{k,l,s=1}^{N}$ are replaced by $\sum_{k=1,l>k}^{N}, \sum_{k=1,l>k,s>l}^{N}$) [40]. It means that instead of $2N_c$ computations, it is sufficient to perform only N_c computations in an easier way.

D. Model Flexibility

The proposed method is both efficient in reducing the computational burden and also very convenient for real life power system operators. As stated earlier, Eqs. (3) can be implemented by symbolic method or by defining the expressions for those derivatives in advance. We assume that symbolic approach is avoided for being slow and one is able to painstakingly define those derivatives for a particular power system. Assuming for the same power system, the type of exciter used changes or a static var compensator (SVC) is introduced, or a PE converter integrated or other changes that introduce modifications of the model, then definitions of (3) have to be extended to accommodate the new models. One has to repeat the process for the added model by defining the derivatives of the new variables with respect to the old ones and vice versa, in order to study these changes. Unfortunately, the above changes are typical of real power system.

Our method acts only on the output of the overall system and the eigenvalues and eigenvectors can be computed by many commercial software used for time simulations. It is not necessary to obtain the derivative of the new model, hence, the method can easily adapt to changes in models. The method is described in the following section.

III. Proposed Method

A. Description

In this section, a detailed explanation of the proposed method is systematically presented. We first consider the static part of (1) and we assume that we are able to compute $\mathbf{f}(\mathbf{X})$ for any $\mathbf{X}$. Up to third order terms, it can be written:

$$\mathbf{P_T} = \mathbf{f}(\mathbf{X_0} + \mathbf{x}) \approx \mathbf{f}(\mathbf{X_0}) + \mathbf{A}\mathbf{x} + \boldsymbol{\beta}(\mathbf{x}). \tag{16}$$

Then, for a given SEP $\mathbf{X}_0$ and a given disturbance $\mathbf{x}$, the nonlinear part can be written:

$$\mathbf{P}_{\text{NL}} = \boldsymbol{\beta}(\mathbf{x}) = \mathbf{P_T} - \mathbf{f}(\mathbf{X_0}) - \mathbf{A}\mathbf{x} = \mathbf{P} - \mathbf{A}\mathbf{x}, \tag{17}$$

where the definition of $\mathbf{P}$ (Eq. (3b)) has been used. Then, considering the static part of (6) and (7), the expression of the r-th component of $\mathbf{f}_{\text{NL}}$ can be written:

$$f_{\text{NL}r} = \sum_{j,k=1}^{N} C^r_{jk} y_j y_k + \sum_{j,k,l=1}^{N} D^r_{jkl} y_j y_k y_l = \mathbf{V}_r^T \mathbf{P}_{\text{NL}}, \quad (18)$$

where Eq. (17) has been used to cancel the linear part.

The method proposed here consists of two successive steps:

- First, one has to prescribe a set of displacement vectors $\mathbf{x}$ which are linear combinations of selected eigenvectors $\mathbf{U}_i$ and compute the corresponding nonlinear force vector $\mathbf{P}_{\text{NL}} = \boldsymbol{\beta}(\mathbf{x})$. The above operation is possible as soon as one can compute $\mathbf{P}_T$ or $\mathbf{f}(\mathbf{X})$ in Eq. (1) for a given $\mathbf{X}$. It does not require to solve any nonlinear system, for instance with a Newton-Raphson technique.
- Second, the unknown coefficients C^r_{jk} and D^r_{jkl} are found by solving a linear system of equations for which the second members are known functions of $\mathbf{P}_{\text{NL}} = \boldsymbol{\beta}(\mathbf{x})$ computed at the previous step.

It is worth to notice that any SEP and model can be considered, with Eq. (18).

To illustrate the method, we first consider prescribing the following perturbation of the SEP:

$$\mathbf{x}_1^{\pm} = \pm \alpha_i U_i \quad \Rightarrow \quad \begin{cases} y_i = \alpha_i, \\ y_j = 0 \; \forall j \neq i, \end{cases} \quad (19)$$

where α_i is an arbitrary constant whose value will be addressed hereafter. The second part of the above equation comes from the orthogonality of the eigenmodes and Eq. (5). Prescribing a perturbation α_i on a given eigenmode $\mathbf{U}_i$ leads to set to zero all the other modal coordinates y_j. Considering Eqs. (19),(18), one obtains:

$$\alpha_i^2 C^r_{ii} + \alpha_i^3 D^r_{iii} = \mathbf{V_r^T P}^+_{\text{NL}}, \quad (20)$$

$$\alpha_i^2 C^r_{ii} - \alpha_i^3 D^r_{iii} = \mathbf{V_r^T P}^-_{\text{NL}}, \quad (21)$$

where $\mathbf{P}^{\pm}_{\text{NL}} = \boldsymbol{\beta}(\mathbf{x}_1^{\pm}) = \boldsymbol{\beta}(\pm\alpha_i \mathbf{U}_i)$. As a consequence, coefficients C^r_{ii} and D^r_{iii} are solutions of a linear system whose second member is easily computed with the static part of the initial system $\mathbf{f}(\mathbf{X})$. This linear system can be generally written as $\mathbf{A_c X_c} = \mathbf{B_c}$, where

$$\mathbf{A_c} = \begin{bmatrix} \alpha_i^2 & \alpha_i^3 \\ \alpha_i^2 & -\alpha_i^3 \end{bmatrix}, \mathbf{X_c} = \begin{bmatrix} C^r_{ii} \\ D^r_{iii} \end{bmatrix}, \mathbf{B_c} = \begin{bmatrix} \mathbf{V_r^T P}^+_{\text{NL}} \\ \mathbf{V_r^T P}^-_{\text{NL}} \end{bmatrix}. \quad (22)$$

For the other coefficients such as C^r_{ij}, D^r_{iij} and D^r_{jji}, $\mathbf{x}$ can be as:

$$\mathbf{x} = \boldsymbol{U_i}\alpha_i + \boldsymbol{U_j}\alpha_j, \; \boldsymbol{U_i}\alpha_i - \boldsymbol{U_j}\alpha_j, \; -\boldsymbol{U_i}\alpha_i + \boldsymbol{U_j}\alpha_j$$

A new linear system is obtained with unknown C^r_{ij}, D^r_{iij} and D^r_{jji} and with the previously obtained coefficients C^r_{ii} and D^r_{iii} in the second member. Finally, the last coefficients D^r_{ijk}, $k \neq i \neq j$, $\mathbf{x}$ are obtained by prescribing:

$$\mathbf{x} = \boldsymbol{U_i}\alpha_i + \boldsymbol{U_j}\alpha_j + \boldsymbol{U_k}\alpha_k. \quad (23)$$

As discussed above, this procedure is very efficient since no Taylor expansion is involved and since it reduces to (i) the evaluation of $\mathbf{f}(\mathbf{X})$ for a given $\mathbf{X}$ and (ii) the solving of linear systems. Another amazing feature of this method is that if we have good information regarding the coefficients of interest, they can be selectively computed without having to do the whole computation. In general, all that are needed are the state matrix which is readily available in many commercial SSA software, and the exact nonlinear equations that model the system.

With (12), the NF coefficients are computed easily by direct substitution. Considering section II-A, step 2 of the classical procedure is totally avoided. The next challenging step is step 6 which is not directly dealt with in this work. Note that the first steps (linear analysis) of our method can be obtained with a commercial software and the other steps can also be implemented (work in progress).

B. Choosing the amplitude of the modal deviation

Although the amplitude of the deviation α is chosen arbitrarily, it should neither be too small nor too big. Very small value of α does not trigger the nonlinearity very well; hence, the system is more or less linear. On the other hand, too large value of α leads to higher nonlinearity, hence the domain of validity of 3rd order approximation is exceeded. Fig.2 shows the sensitivity of α for the system demonstrated in subsection III-C. The modulus of the complex coefficients is shown as a function of α. We observe that the coefficients

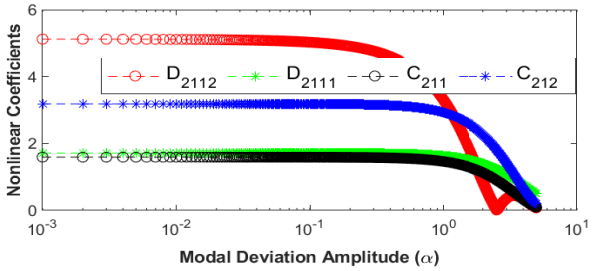

Fig. 2. Sensitivity of Modal deviation amplitude (α)

are consistent for α within certain range. As α becomes large, the nonlinearity increases beyond the validity of the third order approximation. This is evident from Fig.2 as the curves deviate from the actual results. In all cases we have tried, a value in the range of $0.001 \leq \alpha \leq 0.9$ seems to give good result. However, it is necessary to know in advance, the actual value of α that gives the correct result for any given system. The modal deviation amplitude α is now chosen empirically but one major perspective of the work is to be able to define its value in advance considering the degree of nonlinearity of the studied system.

C. Demonstration on a SMIB System

We consider a two degrees of freedom system verifying the following equations:

$$M\dot{\omega} + \frac{EV}{X_T}\sin\delta + D\omega = P_T \quad (24)$$

$$\dot{\delta} - \omega_s \omega = P_T^* \quad (25)$$

M and D are respectively the inertia and damping constants. X_T is the total reactance between the machine and infinite bus, P_T is the mechanical input power while P_T^* in (25) is a dummy parameter which can be zero or not depending on the adopted generator model. For a SMIB system P_T^* is zero at SEP since $\omega_s\omega$ is zero. After perturbation, P_T^* will not necessarily be zero since $\omega_s\omega$ is no longer zero. As P_T is not constant during perturbation in our method, P_T^* is also not constant.

As illustration of how the modal deviation can be applied, assume the system parameters to be: $M = 1, E = 1.123, V = 0.995, X_T = 0.95, D = 2,\ SEP = [0.3 \text{ rad}, 0], P_{T_0} = 0.35$. The right eigenvector is computed as:

$$\mathbf{U} = \begin{bmatrix} -0.0026 + 0.0544i & -0.0026 - 0.0544i \\ 0.9985 + 0.0000i & 0.9985 + 0.0000i \end{bmatrix}$$

The necessary deviations are defined as:

$$\mathbf{x_1^{\pm}} = \pm 0.1 * \begin{pmatrix} -0.0026 + 0.0544i \\ 0.9985 + 0.0000i \end{pmatrix}$$

$$\mathbf{x_2^{\pm}} = \pm 0.1 * \begin{pmatrix} -0.0026 - 0.0544i \\ 0.9985 + 0.0000i \end{pmatrix}$$

In this work, we refer to the conventional method where Taylor and Hessian derivatives are performed as the Direct Hessian method. The $r-$th quadratic and cubic coefficients of SMIB system for the conventional and proposed method are presented in Table I and Table II with h and p denoting the Hessian and the proposed method respectively. As can be seen from the tables, the two results are in agreement.

TABLE I
QUADRATIC COEFFICIENTS

r^{th}	$C_{h_{11}}$	$C_{h_{12}}$	$C_{h_{22}}$	$C_{p_{11}}$	$C_{p_{12}}$	$C_{p_{22}}$
1	-1.59j	-3.18j	-1.59j	-1.59j	-3.18j	-1.59j
2	1.59j	3.18j	1.59j	1.59j	3.18j	1.59j

TABLE II
CUBIC COEFFICIENTS

r^{th}	$D_{h_{111}}$	$D_{h_{112}}$	$D_{h_{221}}$	$D_{h_{222}}$	$D_{p_{111}}$	$D_{p_{112}}$	$D_{p_{221}}$	$D_{p_{222}}$
1	-1.71j	-5.14j	-5.14j	-1.71j	-1.71j	-5.14j	-5.14j	-1.71j
2	1.71j	5.14j	5.14j	1.71j	1.71j	5.14j	5.14j	1.71j

In the next section, the proposed method will be applied to the IEEE 3-Machine, the IEEE 16-Machine, and the IEEE 50-Machine systems. In all cases, the number of coefficients becomes too large to show due to lack of space in this paper. We will focus on the computation time and memory consumption. However, the accuracy will be investigated by checking the maximum error compared to the direct Hessian approach (where possible) which is used here as standard. The modal deviation amplitude (α) of 0.1 was used in all cases.

IV. APPLICATION TO POWER SYSTEMS

In this section we will progressively investigate the application of the proposed method to the three test systems above. In all cases, the state matrix employed by the proposed method was obtained from a commercial software. It is important to note that the complexity arises from the number of state variables (linear basis size) in the system and not from the size of network. Clearly, it is possible for a small network to have a very large computational burden due to the number of state variables. One power electronic (PE) converter for instance can have up to 13 state variables [49] which is computationally larger than 6-machine system with classical model.

A. Test on IEEE 3-Machine System

We first investigate the proposed method on IEEE 3-Machine system shown in Fig. 3. Two-axis model was used with each machine equipped with simple AVR of gain 20. With G3 used as reference the linear basis size is 20, which from (15) amounts to 35,000 NF coefficients . The procedures discussed in section III were followed to compute all the coefficients.

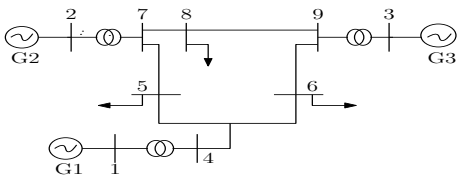

Fig. 3. IEEE 3-Machine System

The maximum errors in percentage were computed as ξ_2= $\max|\frac{C2^j_{kl_{\mathbf{h}}} - C2^j_{kl_{\mathbf{P}}}}{C2^j_{kl_{\mathbf{h}}}}| \times 100$ and $\xi_3 = \max|\frac{D3^j_{pqr_{\mathbf{h}}} - D3^j_{pqr_{\mathbf{P}}}}{D3^j_{pqr_{\mathbf{h}}}}| \times 100$ for quadratic and cubic coefficients respectively. These errors were estimated to be $2e^{-5}\%$ and $1.8e^{-3}\%$ for quadratic and cubic coefficients respectively. The error distribution taking into account all quadratic and cubic coefficients is shown with the red line (Case-A) in Fig.7.

A further comparison by numerical simulation is provided by Fig. 4-5. A 3-phase fault was applied to bus 4 and cleared in 0.184 s. This clearing time is almost the critical clearing time[1] and was chosen to ensure a very severe condition where there should be significant difference in the methods. Then a 3rd order NF model was built using the conventional Hessian and the proposed methods. As expected, the NF solutions show some remarkable deviation from the exact solution due to the severity of the stress. However, the results from the proposed method and the conventional Hessian approach are same which confirms that the proposed method reduces the computational burden without the accuracy compromised.

To validate the method, further investigation was performed on relatively larger systems in the next subsections.

B. Test on IEEE 16-Machine System

The New England/New York System [50] is a widely used 5-Area system consisting of 16 generators, 68 buses and 83 lines. The classical model was used with machine 16 as a reference, making the total number of differential equations 31 and total of 186,434 C and D coefficients.

[1]The critical clearing time is 0.185 s which was obtained by several simulation runs.

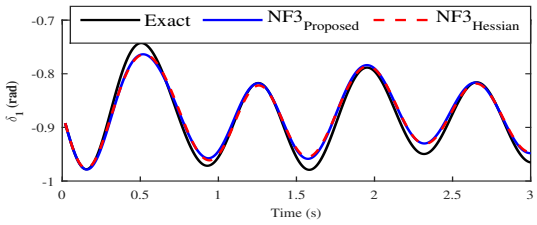

Fig. 4. Machine-1 Relative Angle

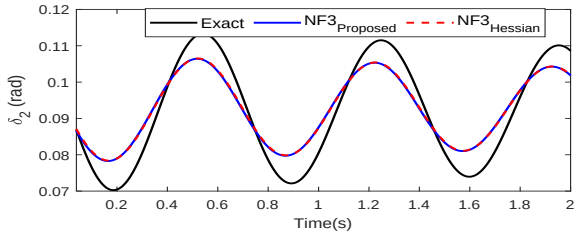

Fig. 5. Machine-2 Relative Angle

The maximum errors were estimated to be $1.7e^{-7}\%$ and $7.3e^{-5}\%$ for quadratic and cubic coefficients respectively. The error here is very infinitesimal compared to the errors in the previous subsection (see case-B in Fig.7). The reason, is likely due to degree of nonlinearity. When there is control in the system such as AVR and PSS, the nonlinearity increases compared to that of classical model. In general, the error incurred by the proposed method is small and is a good compromise for the time and memory saved.

Again, time-domain solution Fig. 6 for a 0.02s short-circuit at bus 53, coming from models built by both the proposed and the conventional Hessian methods show the accuracy of both methods to be same. Note that this system is unstable when no

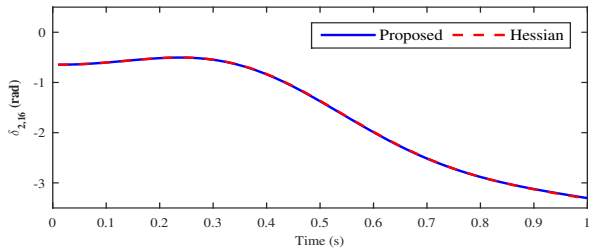

Fig. 6. Machine-2 Relative Angle

PSS or only one PSS is installed [50]. As we did not consider PSS, the system is unstable; the interest in Fig. 6 is only to show the accuracy of the proposed method compared with the standard Hessian approach even for unstable system.

C. Test on IEEE 50-Machine system

In the previous subsections, we have verified the accuracy of the proposed method by comparing with symbolic computation. Here we focus on the number of coefficients and the time for their computation. We applied our method to the IEEE 50-Machine system modelled as classical, with machine 50 taken as reference. There are 16,988,400 C and

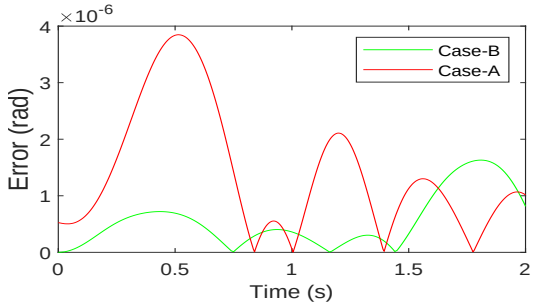

Fig. 7. Error distribution

D coefficients which were computed in approximately 4 hours. Table III shows some selected coefficients. All the coefficients were computed. The presented coefficients were only selected randomly. Fig.8a-8d show the absolute value distributions of C and D coefficients for some modes. Fig.8a-8b is a complete plot of all the C and D coefficients of the first row in Table III (mode 1), while Fig.8c-8d is a complete plot of all the C and D coefficients in row 70 (mode 70), not shown in Table III. It is evident from the figures that both numerically significant and insignificant values are computed.

TABLE III
QUADRATIC & CUBIC COEFF.PROPOSED METHOD

r^{th}	$C_{1,1}$	$C_{10,50}$	$D_{1,60,80}$	$D_{90,90,90}$
1	9.6e-3j	- 1.6e-3j	-2e-4 + 6.1e-3j	9e-6 - 3e-4j
45	1e-4 - 11.9e-3j	2e-4 - 7e-4j	-1e-5 - 1e-4j	3e-5 + 1e-4j
99	-1e-3 + 3e-3j	2e-5 - 2e-4j	2e-4 - 2.5e-3j	11.9e-3j

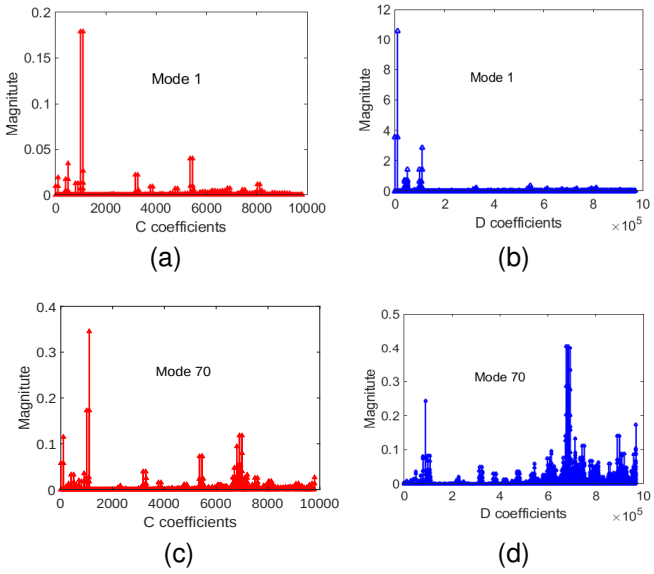

Fig. 8. Distribution of C and D coefficients

In addition to huge computational success already brought by the proposed method, another major advantage is that it allows with convenience, the application of NF to power system focusing on only some selected terms. We note that

to reduce NF computational burden, earlier researches had suggested that higher order spectra (HOS) or prony analyses be used to detect the interacting modes and then only terms relating to these modes can be selectively computed [6], [51]. With the proposed method, any terms of interest can be easily computed. Table III and Fig.8 lend more credence to the fact that some terms can be negligible in NF application. With the proposed method one can reduce the computation burden drastically by neglecting some terms due to the particular modes excited. Certainly, there are some coefficients in NF application that can be neglected even if we are not precise now and in pursuance of such selective NF applications, the proposed method is apt.

D. Computational Efficiency

For the 3-machine system, the time and memory costs were investigated on an Intel CoreTM i7-3520M 2.9GHz laptop computer, and compared with computation using the Symbolic Math Toolbox in MATLAB. The proposed method achieved time and memory saving factors of 43 and 49 respectively (please see columns 8-9 of Table IV). Saving factor is computed as the ratio of the symbolic time/memory consumption to that of the proposed method. Similar comparison for the 16-machine system yielded time and memory saving factors of 472 and 776 respectively (see columns 8-9 of Table IV). In comparison with the symbolic method, this method is indeed a huge success and good news for power system researchers interested in Normal Form based analysis. For further assessment of the computational efficiency, we investigated the time and memory consumption on various system sizes. Fig.9 and Fig.10 show the evolution of computation time and memory consumption for varying size of the system respectively. We observed that for very small size (≤ 5), both methods give fairly comparable results in memory consumption but the times required are remarkably different. Beyond 5 state variables, symbolic method's consumption rate increases tremendously. A summary of some tested cases is shown in Table IV. N/A

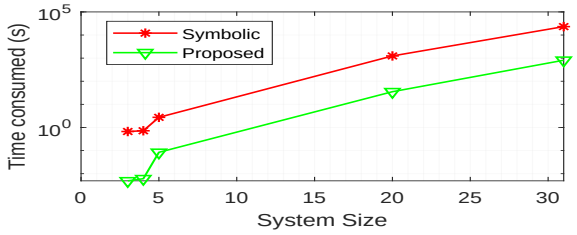

Fig. 9. Computational Time

in Table IV means *not applied.*

It is important to note at this point that the achieved efficiency in this work is limited by the author's programming competence. The algorithm is linear and it is possible to achieve very high efficiency if the code is well optimized. Although we could not compare with the method that involves building the Hessian matrix by predefined derivatives, the proposed method is envisaged to compare favourably with it if optimized. In the memory estimation for symbolic method, we did not consider the sparsity of the DAEs which will of course bring some reduction in memory usage while building the Hessians. However, the modal model (7) which has enough computational burden is not sparse even with sparse DAEs. This still puts the proposed method very far from symbolic method. AD could also be a strong competitor with the proposed method. Although we could not implement AD, a cursory look on both algorithms seems to suggest that our method will likely save more memory under similar program optimization. Firstly, AD computes the nonlinear solutions and its source code simultaneously computes the derivatives, while the proposed method performs only the same nonlinear solution at this first level. The additional memory for the 2nd and 3rd order derivatives may be significant. At the second level, with the derivatives computed by AD, operation (7) has to be done, and separately for 2nd and 3rd order terms. The proposed method obtains directly, the 2nd and 3rd coefficients in (7) in a linear way without any derivative.

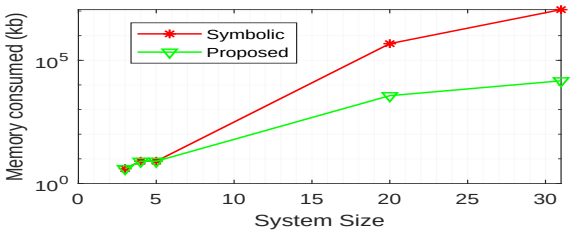

Fig. 10. Computational Memory

V. Conclusions

In this paper we have shown the possibility and method for accelerating NF computation in power systems. The proposed method is an extension of a technique previously employed in structural analysis. In contrast to the conventional method, it avoids Taylor expansion which reduces time and memory in Normal Form computations. The method was explored on SMIB, IEEE 3-Machine, 16-Machine and 50-Machine systems and the results are promising. In comparison with symbolic method, the proposed method proves to save significantly, the computation time and memory.

The method is attractive, in that both second and third order coefficients needed for the NF model are simultaneously evaluated in a linear way and any preferred coefficient can be computed selectively. Many nonlinear modal analyses, such as nonlinear interactions, stability assessments, and nonlinear participation factors usually focus on low frequency modes instead of all modes. This makes the proposed method very useful for quick nonlinear assessments. Also, since the method builds only on the same parameters used for linear analysis, it can easily be integrated in commercial modal analysis software. Moreover, the modal deviations described by our method correspond to initializing a nonlinear system with chosen condition and investigating its solution in steady state. Therefore using same software for linear analysis to achieve NF should be achievable in future.

The presented method is limited to second and third order approximation but the idea can be extended to higher order approximations if necessary. A key limitation of the proposed

TABLE IV
COMPUTATION EFFICIENCY FOR THE TESTED CASES

System	No. of Coeff.	Symbolic		Proposed				
		Time(s)	Memory (Mb)	Time (s)	Memory (Mb)	Max. Error (%)	MSF*	TSF**
IEEE 3-Machine	35,000	1245	475	29	10	1.8e-3	49	43
IEEE 16-Machines	186,434	23,130	11.4e6	49	14.7e3	7.3e-5	776	472
IEEE 50-Machines	16,988,400	N/A	N/A	15,355	1,535	–	–	–

*MSF =Memory saving factor; **TSF = Time saving factor.

method is the choice of the modal deviation amplitudes. It is necessary in future to define the amplitude of the modal deviation as a function of the system parameters instead of heuristically. Further investigation in future is recommended on highly nonlinear grid, such as that with high penetration of power electronic (PE) converters. The challenge of computing Normal Form initial condition is still an open problem and this method does not significantly improve it, although, the method increases the sparsity of all higher order matrices. A perspective of the work is to investigate criteria for determining relevant terms for NF analysis. Such criteria, if found will widen the range of application of the proposed method by again reducing the coefficients.

APPENDIX

DETAILS OF PRESCRIPTIONS IN SECTION III-A

To compute all C^r_{ij}, D^r_{iij} and D^r_{jji}, $\mathbf{x}$ can be as:

$$\mathbf{x} = \boldsymbol{U_i}\alpha_i + \boldsymbol{U_j}\alpha_j,\ \boldsymbol{U_i}\alpha_i - \boldsymbol{U_j}\alpha_j,\ -\boldsymbol{U_i}\alpha_i + \boldsymbol{U_j}\alpha_j \quad \text{(A.1)}$$

Then using (18) we get a set of linear equation

$$\mathbf{A_c}\mathbf{X_c} = \mathbf{B_c} \quad \text{(A.2)}$$

with:

$$\boldsymbol{A_c} = \begin{bmatrix} \alpha_i\alpha_j & \alpha_i^2\alpha_j & \alpha_i\alpha_j^2 \\ -\alpha_i\alpha_j & -\alpha_i^2\alpha_j & \alpha_i\alpha_j^2 \\ -\alpha_i\alpha_j & \alpha_i^2\alpha_j & -\alpha_i\alpha_j^2 \end{bmatrix}, \boldsymbol{X_c} = \begin{bmatrix} C^r_{ij} \\ D^r_{iij} \\ D^r_{jji} \end{bmatrix},$$

$$\boldsymbol{B}_c = \begin{bmatrix} \boldsymbol{V_r}\boldsymbol{P}^{++}_{NL_{ij}} - C^r_{ii}\alpha_i^2 - D^r_{iii}\alpha_i^3 - C^r_{jj}\alpha_j^2 - D^r_{jjj}\alpha_j^3 \\ \boldsymbol{V_r}\boldsymbol{P}^{+-}_{NL_{ij}} - C^r_{ii}\alpha_i^2 - D^r_{iii}\alpha_i^3 - C^r_{jj}\alpha_j^2 + D^r_{jjj}\alpha_j^3 \\ \boldsymbol{V_r}\boldsymbol{P}^{-+}_{NL_{ij}} - C^r_{ii}\alpha_i^2 + D^r_{iii}\alpha_i^3 - C^r_{jj}\alpha_j^2 - D^r_{jjj}\alpha_j^3 \end{bmatrix}$$

Note that all self-coupled coefficients have been previously determined with Eqs. (20) and (21).

For coupling such as $D^r_{ijk}, k \neq i \neq j$, $\mathbf{x}$ can be written as:

$$\mathbf{x} = \boldsymbol{U_i}\alpha_i + \boldsymbol{U_j}\alpha_j + \boldsymbol{U_k}\alpha_k. \quad \text{(A.3)}$$

Then all D^r_{ijk} can be obtained from

$$\begin{aligned} &C^r_{ii}\alpha_i^2 + D^r_{iii}\alpha_i^3 + C^r_{jj}\alpha_j^2 + D^r_{jjj}\alpha_j^3 + C^r_{kk}\alpha_k^2 + \\ &D^r_{kkk}\alpha_k^3 + C^r_{ij}\alpha_i\alpha_j + D^r_{iij}\alpha_i^2\alpha_j + D^r_{ijj}\alpha_i\alpha_j^2 + \\ &C^r_{ik}\alpha_i\alpha_k + D^r_{iik}\alpha_i^2\alpha_k + D^r_{ikk}\alpha_i\alpha_k^2 + C^r_{jk}\alpha_j\alpha_k + \\ &D^r_{jjk}\alpha_j^2\alpha_k + D^r_{jkk}\alpha_j\alpha_k^2 + D^r_{ijk}\alpha_i\alpha_j\alpha_k = \boldsymbol{V_r}\boldsymbol{P}^{+}_{NL_{ijk}}. \end{aligned} \quad \text{(A.4)}$$

Equation (A.4) contains only one unknown term D^r_{ijk} since all other terms have been previously determined.

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