Effective Reynolds Model Coefficients for Flow Between Rough Surfaces in Sliding Motion

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Abstract
In this Letter, it is shown how the determination of the effective coefficients involved in the macroscopic model for pressure driven and/or Couette flow in a rough fracture can be simplified by solving only one closure problem instead of two as originally reported in Prat et al. (Transp Porous Media 48(3):291–313, 2002. https://doi.org/10.1023/a:1015772525610).

Keywords Flow in fractures · Reynolds equation · Upscaling

1 Introduction

In the article "Averaged Reynolds equation for flows between rough surfaces in sliding motion" by Prat et al. (2002), a macroscopic Reynolds model was reported for incompressible, Newtonian and creeping flow within a rough fracture resulting from pressure driven and/or Couette effects (see Fig. 1a). This average model reads (notations from reference Prat et al. (2002) are kept)

$$\nabla \cdot \langle \mathbf{q} \rangle + \mathbf{U}_1 \cdot \langle \nabla h_1 \rangle - \mathbf{U}_2 \cdot \langle \nabla h_2 \rangle = 0, \tag{1a}$$

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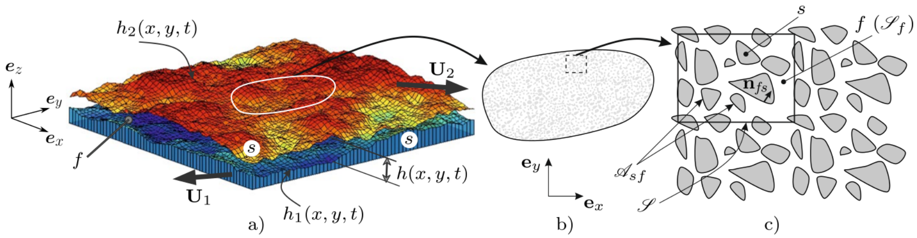

Fig. 1 Sketch of a fracture between two rough surfaces and notations (in agreement with Prat et al. (2002)). **a** Global configuration. **b** Top view of part of the fracture. **c** Representative periodic unit cell, $\mathscr{S}$, in the mid-plane of the fracture including contact spots in gray

$$\langle \mathbf{q} \rangle = -\frac{1}{12\mu}\mathbf{K}^* \cdot \nabla \langle p \rangle^f + \mathbf{C} \cdot \frac{\mathbf{U}_2 - \mathbf{U}_1}{2} + \frac{\mathbf{U}_2 + \mathbf{U}_1}{2}\langle h \rangle. \tag{1b}$$

Here, $\langle \mathbf{q} \rangle$ is the average flow rate per unit width, $\mathbf{U}_i$ ($i = 1, 2$) are the rigid-body velocities of surfaces 1 and 2 located at $h_i = h_i(x, y, t)$ forming the fracture of local aperture $h = h_2 - h_1$, if $h_2 \geq h_1$ and $h = 0$ if $h_2 < h_1$ (i.e., the contact zones) (Prat et al. 2002). In addition, $\langle \psi \rangle$ and $\langle \psi \rangle^f$ denote the superficial and intrinsic averages of a quantity ψ taking values in the fluid phase. They are, respectively, defined as $\langle \psi \rangle = \frac{1}{S}\int_{\mathscr{S}_f} \psi\, dS$ and $\langle \psi \rangle^f = \frac{1}{S_f}\int_{\mathscr{S}_f} \psi\, dS$, where $\mathscr{S}$ (of measure S) is the (two-dimensional) averaging domain of the aperture field, taken as a representative periodic unit cell (see Fig. 1b, c), whereas $\mathscr{S}_f$ (of measure S_f) is the subdomain of $\mathscr{S}$ occupied by the fluid phase, which excludes the contact zones (where $h = 0$). Finally, in Eq. (1b), $\mathbf{K}^*$ and $\mathbf{C}$ are two second-order tensors, respectively, representing the effective transmissivity and Couette-effect coefficient. They are defined as

$$\mathbf{K}^* = \langle K(\mathbf{I} + \nabla \mathbf{b}) \rangle, \tag{2}$$

$$\mathbf{C} = \langle K \nabla \mathbf{c} \rangle. \tag{3}$$

Here, $K = h^3$ and $\mathbf{I}$ is the identity tensor, whereas the two vectors $\mathbf{b}$ and $\mathbf{c}$ are the closure variables that map the influences of the macroscopic sources, namely $\nabla \langle p \rangle^f$ and $\mu(\mathbf{U}_2 - \mathbf{U}_1)/6$, onto the spatial deviations of the fluid pressure. In other words, $\mathbf{b}$ and $\mathbf{c}$ are the local multipliers applied to the corresponding sources providing the expression of the pressure deviation (see equation (16) in Prat et al. (2002) in which $\mathbf{c}_1 = -\mathbf{c}_2 = 6\mathbf{c}$ and $\varphi = 0$). Formally, these variables can be shown to be defined as the integrals of the Green's functions associated to the flow problem in a periodic representative unit cell of the fracture (see Fig. 1c). They solve the following two independent closure problems (see equations (21) and (22) in Prat et al. (2002))

Problem I

$$\nabla \cdot (K(\nabla \mathbf{b} + \mathbf{I})) = \mathbf{0}, \quad \text{in } \mathscr{S}_f, \tag{4a}$$

$$\mathbf{b}(\mathbf{r} + \mathbf{l}_i) = \mathbf{b}(\mathbf{r}), \quad i = x, y, \tag{4b}$$

$$\langle \mathbf{b} \rangle^f = \mathbf{0}. \tag{4c}$$

Problem II

$$\nabla \cdot \left(K \nabla \mathbf{c} - 2h_+ \mathbf{I} \right) = \mathbf{0}, \quad \text{in } \mathscr{S}_f, \tag{5a}$$

$$\mathbf{c}(\mathbf{r} + \mathbf{l}_i) = \mathbf{c}(\mathbf{r}), \quad i = x, y, \tag{5b}$$

$$\langle \mathbf{c} \rangle^f = \mathbf{0}. \tag{5c}$$

In these equations, $\mathbf{r}$ denotes the coordinate of a point within $\mathscr{S}_f$, $\mathbf{l}_i$ is the periodic lattice vector in the ith direction ($i = x, y$), and $h_+ = (h_1 + h_2)/2$ is the mean surface. Note that, since the axis origin for h_1 and h_2 is arbitrary, it can be taken such that $h_1 = h_2 = 0$ at the mid-plane of the contact, which implies that $h_+ = 0$ at the contours, $\mathscr{A}_{sf}$, of the contact zones in $\mathscr{S}$ (see Fig. 1c). The reader is referred to Prat et al. (2002) for the details on the derivation of the above two problems.

It must be noted that, although not stated in the original article, the following boundary conditions at $\mathscr{A}_{sf}$ must be included in the two closure problems if contact zones are present

$$\mathbf{n}_{fs} \cdot (\nabla \mathbf{b} + \mathbf{I}) = \mathbf{0}, \quad \text{at } \mathscr{A}_{sf}, \tag{6}$$

$$\mathbf{n}_{fs} \cdot \nabla \mathbf{c} = \mathbf{0}, \quad \text{at } \mathscr{A}_{sf}. \tag{7}$$

Here, $\mathbf{n}_{fs}$ ($= -\mathbf{n}_{sf}$, see Fig. 2 in Prat et al. (2002) and Fig. 1c) is the unit normal vector at $\mathscr{A}_{sf}$ pointing out of $\mathscr{S}_f$. Along with these last two conditions, problems I and II, that are intrinsic for a given fracture at a given time, are well-posed and have a unique solution.

The objective of this Letter is to show that the two effective coefficients $\mathbf{K}^*$ and $\mathbf{C}$ given in Eqs. (2) and (3) can be obtained from the solution of only closure Problem I. This is desirable since, in this way, the computational time is reduced by a factor of 2 and this can benefit practical applications such as the prediction of leakage rate of seals, gas recovery in fractured rocks, lubrication processes, among others.

2 Alternative expressions for C

The proof that it is not necessary to solve closure Problem II starts by considering the following integral formula that relies on Green–Ostrogradski's theorem applicable for any arbitrary second-order tensor field, $\mathbf{A}$, and vector field, $\mathbf{a}$, taking values in a domain Ω_β of boundary $\partial\Omega_\beta$, and having appropriate regularity. This formula reads

$$\int_{\Omega_\beta} \nabla \cdot (\mathbf{A}\mathbf{a})\, d\Omega = \int_{\Omega_\beta} (\nabla \cdot \mathbf{A})\, \mathbf{a}\, d\Omega + \int_{\Omega_\beta} \mathbf{A}^T \cdot \nabla \mathbf{a}\, d\Omega = \int_{\partial\Omega_\beta} \mathbf{n} \cdot \mathbf{A}\, \mathbf{a}\, dS. \tag{8}$$

In this expression, $\mathbf{n}$ is the unit normal vector at $\partial\Omega_\beta$ pointing outside Ω_β.

The above identity can now be considered in the periodic unit cell, $\mathscr{S}$, representative of the fracture ($\Omega_\beta \equiv \mathscr{S}_f$) for $\mathbf{A}$ and $\mathbf{a}$ being periodic and with $\mathbf{A}$ satisfying $\nabla \cdot \mathbf{A} = \mathbf{0}$ and $\mathbf{n}_{fs} \cdot \mathbf{A} = \mathbf{0}$ at $\mathscr{A}_{sf}$. Under these circumstances, Eq. (8) leads to

$$\langle \mathbf{A}^T \cdot \nabla \mathbf{a} \rangle = \mathbf{0}. \tag{9}$$

This relationship is now employed taking $\mathbf{A} = K(\nabla \mathbf{b} + \mathbf{I})$ and $\mathbf{a} = \mathbf{c}$ to obtain

$$\langle K \nabla \mathbf{c} \rangle = -\langle K \nabla \mathbf{b}^T \cdot \nabla \mathbf{c} \rangle. \tag{10}$$

An additional use of the integral identity in Eq. (9) can be made with $\mathbf{A} = K\nabla \mathbf{c} - 2h_+ \mathbf{I}$ and $\mathbf{a} = \mathbf{b}$. When the transpose of the resulting equation is taken, this yields

$$\langle K \nabla \mathbf{b}^T \cdot \nabla \mathbf{c} \rangle = 2\langle h_+ \nabla \mathbf{b}^T \rangle. \tag{11}$$

Substitution of this last result back into Eq. (10), taking into account the definition of $\mathbf{C}$ in Eq. (3), gives the following expression of this effective coefficient in terms of the closure variable $\mathbf{b}$

$$\mathbf{C} = -2\langle h_+ \nabla \mathbf{b}^T \rangle. \tag{12}$$

An alternative form of the above expression for $\mathbf{C}$ can be obtained by making use of the averaging theorem (Whitaker 1999), which, in the context of the present study, writes $\langle \nabla \psi \rangle = \nabla \langle \psi \rangle + \frac{1}{S} \int_{\mathscr{A}_{sf}} \mathbf{n}_{fs} \psi \, d\ell$. Taking $\psi = h_+ \mathbf{b}$, this theorem leads to

$$\langle \nabla (h_+ \mathbf{b}) \rangle = \nabla \langle h_+ \mathbf{b} \rangle + \frac{1}{S} \int_{\mathscr{A}_{sf}} \mathbf{n}_{fs} h_+ \mathbf{b} \, d\ell = \nabla \langle h_+ \mathbf{b} \rangle = \langle \nabla h_+ \mathbf{b} \rangle + \langle h_+ \nabla \mathbf{b} \rangle. \tag{13}$$

However, since both h_+ and $\mathbf{b}$ are considered as periodic fields in closure problems I and II, $\nabla \langle h_+ \mathbf{b} \rangle = \mathbf{0}$, and hence $\langle h_+ \nabla \mathbf{b} \rangle = -\langle \nabla h_+ \mathbf{b} \rangle$. Substituting this result in Eq. (12) finally provides the following alternative expression for $\mathbf{C}$

$$\mathbf{C} = 2\langle \nabla h_+ \mathbf{b} \rangle^T = 2\langle \mathbf{b} \nabla h_+ \rangle. \tag{14}$$

Equations (12) and (14) show that $\mathbf{C}$ can be obtained from the solution of closure Problem I.

Using a similar approach to that leading to Eq. (14), an alternative expression for $\mathbf{K}^*$ can be obtained that writes

$$\mathbf{K}^* = \langle K \mathbf{I} \rangle - \langle \nabla K \mathbf{b} \rangle. \tag{15}$$

Expressions (2) and (12) for, respectively, $\mathbf{K}^*$ and $\mathbf{C}$ may nevertheless be preferred since they only involve $\nabla \mathbf{b}$. Indeed, with these forms, the field of $\mathbf{b}$ can be determined to within an arbitrary additive constant, thus relaxing the constraint indicated in Eq. (4c), that may be replaced by any other convenient one (if necessary) to ease solving closure Problem I.

3 Conclusions

The above derivations demonstrate that both $\mathbf{K}^*$ and $\mathbf{C}$ are obtained from the solution of Problem I. The net outcome is a significant simplification in the closure process of the upscaled model derived in Prat et al. (2002) as the computing requirement is divided by a factor of 2.

The results from this work have a practical significance and can be used following these steps: (i) determine both surface topologies, i.e., h_1 (the bottom one) and h_2 (the top one) on representative areas and form the assembly of the two according to the configuration of interest at $t = 0$; (ii) compute the aperture field $h = h_2 - h_1$, setting $h = 0$ when $h_2 - h_1 < 0$; (iii) determine the mean surface $h_+ = (h_1 + h_2)/2$, taking the mid-plane of the fracture as the origin for both h_1 and h_2 fields; (iv) solve closure Problem I. Methodologies and examples of solution to this problem can be found in Vallet et al. (2009a, 2009b), Zaouter et al. (2018, 2019, 2023; v) compute $\mathbf{K}^*$ from Eq. (2) and $\mathbf{C}$ from Eq. (12); (vi) use Eq. (1b) to predict the flow rate through the fracture, knowing the applied pressure gradient $\nabla\langle p\rangle^f$ (as well as $\mathbf{U}_1$ and $\mathbf{U}_2$); (vii) repeat steps (ii)–(vi) at any desired time at which the relative position of the two surfaces is recomputed knowing $\mathbf{U}_1$ and $\mathbf{U}_2$ to predict the flow rate, $\langle\mathbf{q}\rangle$, at this time.

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Declarations

Conflict of interest The authors report no conflict of interest.

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