The drainage of non-Newtonian fluids in the quasi-steady motion of a sphere towards a plane

A. Despeyroux · A. Ambari

Abstract In the lubrication limit, the time needed for the drainage of the liquid film between two particles or between particles and walls is of industrial importance, because it controls the dynamics and aggregation of non-dilute suspensions. This problem is also of fundamental interest in the application of the dynamic surface force apparatus to nanorheology. Even if this problem has an exact solution in Newtonian fluid when the sphere moves steadily and slowly towards or away from a plane wall, this problem remains, to our knowledge, without any exact analytical solution in non-Newtonian fluids with negligible viscoelastic components. But Rodin, using the method of asymptotic expansions, gives an asymptotic solution to this problem in the lateral unbounded power-law fluid. Therefore, in this study, we give a numerical result using the dynamic mesh technique and an asymptotic analytical formula valid in the lubrication regime, for a fluidity index $0.5 < n \leqslant 1.8$. The comparison between the two results confirms their mutual validity.

Keywords Dynamic surface force apparatus · Non-Newtonian fluids · Power-law fluids · Sphere towards a plane · Hydrodynamic interactions

List of symbols

- a Radius of the sphere, m
- b Radius of the tube, m
- $\underline{\underline{D}}$ Rate of strain tensor, 1/s
- $\mathbf{e}_z$ Unit vector in z-direction, dimensionless
- $\mathbf{F}$ Drag force undergone by the sphere in z-direction, N
- h Minimum distance between the sphere and the plane, m
- k $= a/b$, lateral confinement coefficient, dimensionless
- m Consistency of the fluid, Pa.s^n
- n Fluidity index of the power-law fluid, dimensionless
- p Pressure, Pa
- r Radial coordinate, m
- r_+ $= r/a$, reduced radial coordinate, dimensionless
- Re_n Generalized Reynolds number, dimensionless
- U Sphere velocity, m/s
- z Axial coordinate, m

Greek symbols

- δ Perpendicular correction factor of the drag force undergone by a sphere approaching a plane, dimensionless
- ε $= h/a$, normalized minimum distance between the sphere and the plane, dimensionless
- ε_1 $= a\varepsilon + a(1 - \cos\theta)$, local distance between the sphere and the plane, m
- η Newtonian viscosity, Pa.s
- ν Kinematic viscosity, m^2/s
- θ Angular coordinate, rad
- ρ Fluid density, kg/m^3
- $\underline{\underline{\tau}}$ Stress tensor, Pa

1 Introduction

The interactions between solid particles in dispersions are mainly due to the hydrodynamic drainage processes as long as the distance between the macroscopic surfaces in liquid are above 10–15 nm (Bhushan 2010). In fact, due to the smaller value of the Hamaker constants for molecular interaction in liquid than in vacuum (Bhushan 2010), the Van der Waals forces go into action in a very thin layer (10–15 nm). Outside of this zone, the hydrodynamic

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drainage of the liquid between particles controls their approach until the other molecular forces take over, as in the coagulation of colloids (Hocking 1973; Potanin et al. 1988). In another field, recent applications of dynamic surface force apparatus (DSFA) in nanorheology (Restagno et al. 2002) need the knowledge of this kind of hydrodynamic interaction. The study of the adherence properties of a liquid near a solid interface i.e. the flow boundary condition and the role of hydrophobization of a solid surface on the fluid/wall slippage, is carried out by the use of a DSFA which consists of the measure, in the quasi-steady and oscillatory regime, of the hydrodynamic force undergone by a sphere moving towards one flat wall (Cottin-Bizonne 2003). The use of this DSFA as a nanorheometer is very promising and needs the knowledge of the relation between the hydrodynamic force undergone by a sphere and its displacement towards the plane in the lubrication approximation for Newtonian and non-Newtonian fluids (Cottin-Bizonne 2003; Luengo et al. 1997; Horn et al. 2000). Moreover, the time needed for the contact of two spheres in Newtonian or non-Newtonian fluid plays an important role in the aggregation and the formation of the plug flow during the transport of suspensions. Another application concerns the particle–particle or particle–wall collision in non-Newtonian fluids, which takes place in particle-laden flows (Stocchino and Guala 2005; Guala and Stocchino 2007; Ardekani et al. 2009; Marston et al. 2010). Therefore, this study deals with the calculation of the Stokes-type law correction factor, for the hydrodynamic resistance of a sphere of radius a moving at steady velocity U, towards or away from a plane wall in power-law fluids.

As carried out by the DSFA, we will focus on the case where the moving velocity U is maintained constant during the approach to the rigid plane (at fixed low Reynolds number) in incompressible Newtonian or power-law fluid of a given apparent viscosity, which is dependent only on the strain rate. Under this condition, as the geometry of the problem changes over time, due to the linear variation of the distance $h = \varepsilon a$ between the sphere and the plane wall in time, we assume first that a quasi-steady solution applies, and the added mass force and the history force are irrelevant in this configuration.

The condition of quasi-static flow was expressed by Cox and Brenner (1967) as $\varepsilon Re \ll 1$ where $Re = 2aU/\nu$ and ν the kinematic viscosity. This condition can be found by assuming that the velocity field in the gap induced by the relative motion of the sphere with respect to the wall must establish itself in a characteristic time shorter than the unsteady convective time $\varepsilon a/U$.

In lateral unbounded Newtonian fluid, this correction factor $\delta(\varepsilon)$ has been calculated analytically by Brenner (1961) and Maude (1961). They used bipolar coordinates which had been first employed by Stimson and Jeffery (1926).

$$\mathbf{F}(\varepsilon) = -6\pi\eta a U \delta(\varepsilon)\mathbf{e}_z \qquad (1)$$

The asymptotic expansion of their formula for small gap εa between the plane and the sphere is given by Cox and Brenner (1967):

$$\delta(\varepsilon) = \varepsilon^{-1}\left(1 - \frac{1}{5}\varepsilon \ln\left(\frac{1}{\varepsilon}\right) + 0.9712\varepsilon\right) \qquad (2)$$

which is valid for $\varepsilon \leqslant 0.6$ (see Fig. 3). The first term of this asymptotic expansion is the well-known Taylor law

$$\delta(\varepsilon) = \varepsilon^{-1} \qquad (3)$$

which is valid only for $\varepsilon \leqslant 0.04$ (see Fig. 3). The Brenner formula has been successfully verified experimentally by Ambari et al. (1984b).

However, as far as we know, there are no exact analytical or numerical results concerning the calculation of this perpendicular correction factor of the drag undergone by a sphere translating at constant velocity towards a plane wall in non-Newtonian fluids. In the limit of lubrication, using the asymptotic expansions method, an asymptotic solution is given to this problem by Rodin (1996) in lateral unbounded power-law fluid, for the squeezing motion of two nearly touching rigid spheres (where the case of a sphere moving towards a plane is deduced by setting the radius of one sphere to an infinite value). Then as a first approximation of the non-Newtonian behaviour of the fluid, in this study, we try to give a numerical and asymptotical solution for this problem, using the Reynolds lubrication equations, in power-law fluids whose behaviour can be described by the following constitutive equation. Note that this model constitutes one type of generalized Newtonian fluid whose apparent viscosity depends only on its second invariant of the strain tensor.

$$\underline{\underline{\tau}} = 2\left[m\left(\sqrt{2\underline{\underline{D}} : \underline{\underline{D}}}\right)^{n-1}\right]\underline{\underline{D}} \qquad (4)$$

where $\underline{\underline{\tau}}$ is the stress tensor and $\underline{\underline{D}} = \frac{1}{2}(\nabla\mathbf{U} + \nabla^T\mathbf{U})$ is the strain rate tensor, m the power-law consistency coefficient (Pa s^n) and n the fluidity index of the power-law fluid (called Ostwald–de Waele fluid). For shear flow, Eq. (4) is reduced to $\tau_{xy} = m\dot{\gamma}^n$ where $\dot{\gamma} = \partial u_x/\partial y$ is the shear rate. In this case, the apparent viscosity is given by $m\dot{\gamma}^{n-1}$. When $n < 1$, this apparent viscosity decreases, the fluid is called shear thinning or pseudoplastic; for $n > 1$, the apparent viscosity increases and the fluid is called shear thickening or dilatant. For $n = 1$, the fluid has a Newtonian behaviour. In fact, in this particular case of the calculation of the hydrodynamic drag experienced by a particle moving towards a plane, the force is due to the drainage effect in the lubrication regime. In this condition, the non-

Newtonian behaviour which dominates in this drainage flow may be of a power-law type (for many non-Newtonian fluids). This is due to the weakness of the elongational velocity gradient (induced by the very low moving velocity of the particle in DSFA), which is able to arouse the viscoelasticity of the fluid. In fact, in this work, we consider non-Newtonian fluid with a relaxation time much lower than the inverse of the maximum of the normal and radial elongational velocity gradient (low Deborah numbers); located in the vicinity of the stagnation point, the effect of the viscoelasticity is negligible (de Gennes 1974; Hinch 1974; Ambari 1979). Under this condition, the behaviour of the fluid in this almost Poiseuille-type flow (viscometric one) can be modelled in first approximation by a power-law fluid, which takes account of a possible shear thinning behaviour. Furthermore, recent experimental results on particle–wall collision in polymeric liquids (Guala and Stocchino 2007; Ardekani et al. 2009) confirmed that the viscoelasticity of the fluid was negligible relative to its shear thinning character at low Deborah numbers, supporting the choice of the power-law model in this particular flow. Moreover, as the drag is principally due to the pressure induced by the drainage flow, the shear velocity gradient is limited by its value reached near the axis of symmetry. Indeed, most non-Newtonian fluids exhibit Newtonian plateaus at low and high shear rate, in the evolution of their viscosity with the velocity gradient, and their behaviour can be described by the widely used Carreau–Yasuda model (Carreau et al. 1997). In the present configuration of a sphere approaching a plane in the shear thinning case, as long as the maximum velocity gradient is not included in the possible second Newtonian plateau of the power-law fluid, and as long as the low shear velocity gradient corresponding to the first Newtonian plateau, situated in the extreme vicinity and far from the axis of the sphere, has very little contribution to the calculation of the drag, the Ostwald model can constitute a good approximation. A similar analysis is applicable to shear thickening fluids. Under these conditions, the use of this simple model with only one control parameter which is the fluidity index n, contrary to that of a more realistic model such as Carreau–Yasuda introducing four control parameters (Carreau et al. 1997), enables us to clearly and physically show the influence of the shear thinning and the shear thickening behaviour on the drag undergone by this particle.

2 Formulation and numerical approach

The flow of a fluid around a sphere, of radius a, moving towards a plane at constant velocity $\mathbf{u}_z = -U\mathbf{e}_z$ is shown schematically in Fig. 1. For the needs of the numerical calculations, the sphere moves axially inside a very large cylindrical tube of radius b towards its bottom. This container is filled with a Newtonian or power-law fluid. For simplicity of the calculations, we consider an equivalent situation where it is assumed that the sphere is set and the walls of the cylinder (bottom, top and lateral walls) move at the velocity $U\mathbf{e}_z$. The flow is governed by the usual conservation equations for mass and momentum under isothermal conditions, i.e.

$$\begin{cases} \nabla\cdot\mathbf{U} = 0 \\ \boldsymbol{\rho}[\partial_t\mathbf{U} + (\mathbf{U}\cdot\nabla)\mathbf{U}] = -\nabla p + \nabla\cdot\underline{\underline{\tau}} \end{cases}$$

where ρ is the fluid density, p is the pressure and $\underline{\underline{\tau}}$ is the stress tensor for the power-law fluid (formula 4). The velocity boundary and initial conditions are:

1. On all the walls of the cylinder: $\mathbf{U} = -U\mathbf{e}_z$;
2. On the sphere: $\mathbf{U} = \mathbf{0}$;
3. For $t \leqslant 0$ the fluid is at rest: $\mathbf{U} = \mathbf{0}$.

At very low generalized Reynolds numbers $Re_n = \rho U^{2-n}(2a)^n/m$, the calculation of the wall correction factor $\delta(n, \varepsilon)$ of the drag experienced by a sphere translating towards a plane in the axis of the tube can be expressed by the following expression, which can be obtained by dimensional analysis:

$$\mathbf{F}(n, \varepsilon) = -6\pi m \left(\frac{U}{2a}\right)^{n-1} aU\delta(n, \varepsilon)\mathbf{e}_z \tag{5}$$

where $\varepsilon = h/a$ is the normalized distance between the sphere and the wall. Let us recall that we gave an accurate numerical solution (Despeyroux et al. 2011) to the problem concerning the drag force undergone by a sphere in power-law fluids in unbounded medium, whose results had not been definitively established in non-inertial regime and particularly for dilatant fluids. The polynomial interpolation formula which gives correct values of this coefficient with an average relative error less than 1% for $0 \leqslant n \leqslant 1.8$ is given by:

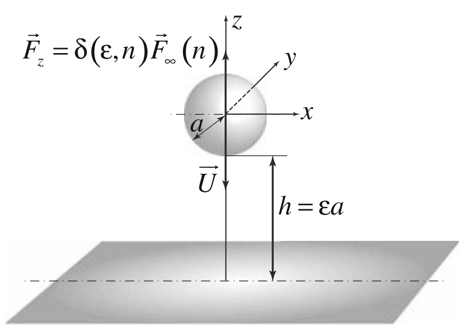

Fig. 1 Geometrical and hydrodynamical parameters for a sphere moving towards a plane

$$\delta(n,\varepsilon=\infty)=1.1909+1.9781n-4.9165n^2+7.3332n^3 \\ -8.7174n^4+5.7425n^5-1.8380n^6+0.22818n^7 \quad (6)$$

To obtain a solution for this problem whose geometry changes over time, it is not possible to use the steady classical numerical method. For this reason, we used the dynamic mesh method in the finite volume CFD FLUENT code where the SIMPLE algorithm was employed with a second-order scheme. These computations are carried out on a structured mesh, and a nonstructured one only in the vicinity of the stagnation point of the sphere to ensure a homogeneous fixed mesh size during the deformation of the mesh. In fact, this dynamic mesh technique implies, for each time step, the rigid motion of the bottom of the container towards the boundary of the sphere. The mesh is then adjusted according to the new position of the moving boundaries. In our case, in the rectangular mesh zone near the bottom, the dynamic layering removes layers of cells adjacent to the moving boundary, based on the height of the layer adjacent to the moving surface. In fact, the cell is split or merged with the layer of cell next to it when its layer attains a critical height. The minimum gap which can be reached through this procedure is the minimum thickness of the nonstructured mesh zone at the stagnation point of the sphere ($\varepsilon = 10^{-3}$). For this computation, we used a 16-core cluster. The convergence of the computation at each step l is supposed to be reached when the following criterion is verified: $\left|1-\delta^l(n,\varepsilon)/\delta^{l+1}(n,\varepsilon)\right|<10^{-6}$. Let us recall that a 100 iterations are used for each time step. Moreover, it is noticeable that as we are concerned only with a bounded shear stress τ_{rz} corresponding to a bounded shear rate $0 \leqslant \dot{\gamma}_{rz} \leqslant \dot{\gamma}_{rzmax}$ in this particular flow, where $\dot{\gamma}_{rzmax}$ is reached all the closer to the minimum gap as the sphere approaches the plane, there is no singularity in the shear stress even if the apparent viscosity described by this model mathematically diverges for zero shear rate. Indeed, as the zones where the velocity gradient is negligible do not introduce a significant contribution to the calculation of the drag, we proceeded to a verification of the effect of a truncated Ostwald model at low and high velocity gradients as in the Carreau–Yasuda model (Carreau et al. 1997), with a variation of the apparent viscosity over three decades in accordance with most rheological experiments (Carreau et al. 1997). Then the variation we imposed was centred around the value corresponding to the mean velocity gradient. As seen in Fig. 7 in Sect. 4.2, the results corresponding to this truncated Ostwald model remain the same as those obtained using the complete Ostwald model. To check the validity of the results obtained by the numerical method employed in this study, we proceeded to their comparison with those obtained by the exact solution in Newtonian fluid and an asymptotic approach in the limit of the lubrication regime in non-Newtonian fluid.

3 Asymptotic approach

When the sphere moves towards the plane, in the limit of the lubrication regime $\varepsilon \ll 1$ (see Fig. 2) and for very low Reynolds numbers, the dissipation would be located principally in the minimum gap remaining between the sphere and the plane. In this configuration, the drag force undergone by a sphere can be calculated from the pressure force induced by the radial drainage flow (Cox 1974; Vinogradova 1995). Otherwise, in this limit, Rodin (1996) gave an asymptotic solution for the squeezing motion of two nearly touching rigid spheres (S_1 of radius a and S_2 of radius β a) in a power-law fluid. To solve this problem, he used the asymptotic expansions of the axisymmetric Stokes stream function and the asymptotic problem was analyzed in nondimensional stretched coordinates. He calculated the solution to the pressure for different values of the $\alpha = (1+\beta)/2\beta$ parameter and deduced the drag submitted by each sphere by integrating the pressure transmitted by a horizontal circle of radius a centred at the origin. Concerning our configuration of the sphere settling towards a plane, we took $\beta \to \infty$ then $\alpha = 1/2$. When we replace this α value in his expression for the pressure, we obtain this radial distribution of the pressure:

$$\frac{p(r_+)-p_\infty}{m\left(\frac{U}{2a}\right)^n} = \left(\frac{2n+1}{n}\right)^n\left(\frac{2^{2+3n}}{1+3n}\right)(r_+)^{-(1+3n)} \\ \times {}_2F_1\left(1+2n, \frac{1}{2}(1+3n); \frac{3}{2}(1+n); -2\frac{\varepsilon}{r_+^2}\right) \quad (7)$$

where $r_+ = r/a$ is the normalized radial distance from the stagnation point, p_∞ the pressure far from the gap and ${}_2F_1$ is the Gaussian hypergeometric function. For $n = 1$ corresponding to the Newtonian fluid, this expression reduces to:

$$\frac{p(r_+)-p_\infty}{\mu\left(\frac{U}{2a}\right)} = \frac{6}{\left[\varepsilon+\frac{1}{2}r_+^2\right]^2} \quad (8)$$

which is the same as the classical lubrication solution (Pasol et al. 2005; Chan and Horn 1985; Mongruel et al. 2011). In this approach, the correction factor $\delta(n, k=0, \varepsilon)$ can also be obtained for $\beta = 1/2$ in the expression of the force given by Rodin (1996):

$$\delta(n>1/3, k=0, \varepsilon) = \frac{F(n, k=0, \varepsilon)}{F(n, k=0, \varepsilon \to \infty)} \\ = \frac{2^{\frac{3n-1}{2}}}{3}\left(\frac{2n+1}{n}\right)^n \\ \times \beta\left(\frac{3+n}{2}, \frac{3n-1}{2}\right)\frac{1}{\varepsilon^{\frac{3n-1}{2}}} \quad (9)$$

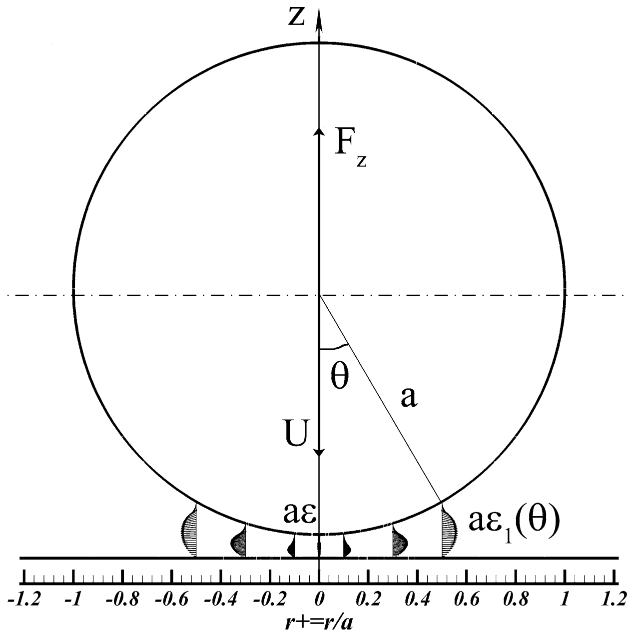

Fig. 2 Numerical velocity field in the gap between the sphere and the plane wall in the lubrication regime ($\varepsilon = 10^{-2}$) for $n = 0.8$

This formula is valid mathematically only for $n > 1/3$. For $n = 1/3$, he obtained:

$$\delta(n = 1/3, k = 0, \varepsilon) = \frac{2^{\frac{3n-1}{2}}}{3} \left(\frac{2n+1}{n}\right)^n \log(\varepsilon^{-1}) \qquad (10)$$

and for $n < 1/3$, he gave:

$$\delta(n < 1/3, k = 0, \varepsilon) = \frac{2^{\frac{3n-1}{2}}}{3} \left(\frac{2n+1}{n}\right)^n O(1) \qquad (11)$$

where $O(1)$ is a constant. These two last results will be discussed in Sect. 4.2. For the Newtonian case, the solution 9 reduces to the classical Taylor solution:

$$\delta(n = 1, k = 0, \varepsilon) = \frac{1}{\varepsilon} \qquad (12)$$

Moreover, we propose in the appendix a simple asymptotic solution, using the Reynolds lubrication equations, corresponding to the sphere moving towards a plane in the limit of the lubrication regime and avoiding the use of the streamfunction. This calculation gives the similar asymptotic solution as obtained from Rodin's formula corresponding to $\beta = \infty$.

4 Results and discussion

Hereafter, we first give a comparison of the numerical and asymptotical results obtained for Newtonian fluid with the aim of giving a validation of the numerical method and asymptotical approach used in this work. In a second step, we give the non-Newtonian correction factor numerically and asymptotically in the case of the power-law fluid for different indexes of fluidity.

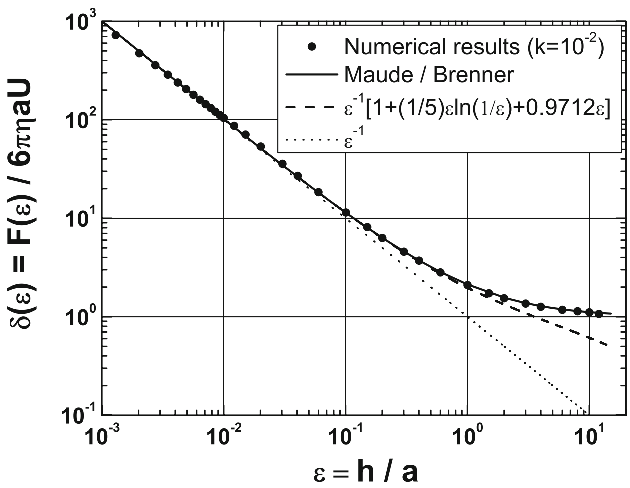

Fig. 3 Validation of the numerical calculation of the effect of the frontal confinement on the drag force undergone by a moving sphere towards a bottom of a tube filled with a Newtonian liquid at lateral confinement $k = 10^{-2}$. Comparison between the analytical and asymptotic results for a lateral unbounded medium

4.1 Newtonian fluid and validation

In the numerical simulation, the sphere has to move towards a plane in a very large cylinder of radius b. Our first concern was the study of the effect of the lateral confinement defined by the ratio $k = a/b$. In fact, Fig. 3 shows the good accordance between the numerical results of the perpendicular correction factor obtained numerically for $k = 10^{-2}$ and those obtained by the exact analytical solution given by Brenner (1961) and Maude (1961) in lateral unbounded medium. This agreement is all the more perfect as the sphere is in the lubrication regime ($\varepsilon \to 0$), and the drag becomes independent from the lateral confinement. In Fig. 4, the radial distribution of the pressure calculated numerically for $k = 10^{-2}$ and $\varepsilon = 10^{-2}$, at low Reynolds number ($Re = 10^{-3}$), is also in good agreement with that calculated asymptotically for lateral unbounded medium and given by the formula 8.

Finally, we note the accuracy of the slope of the power-law decrease of the pressure with the radial distance $(p(r_+) - p_\infty) \propto r_+^{-4}$. This successful comparison in the Newtonian case supports the validity of the numerical method used in this work.

4.2 Non-Newtonian fluid

First of all, as we consider in this work the case where the possible relaxation time of the fluid is much lower than the

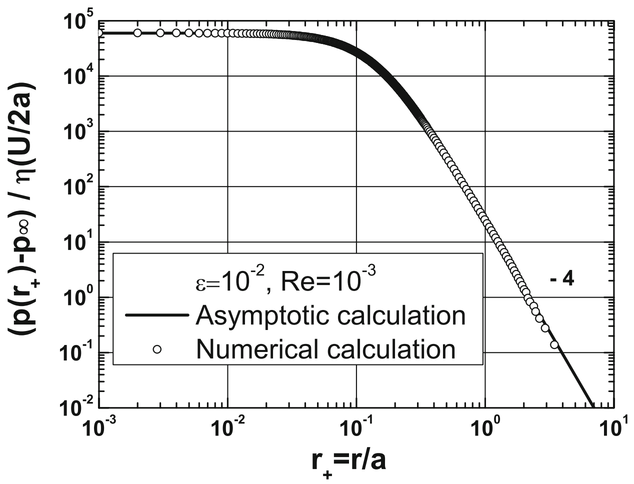

Fig. 4 Comparison of the evolution of the pressure in the gap between the sphere and the plane wall in lubrication regime ($\varepsilon = 10^{-2}$), calculated numerically and asymptotically (formula 8) for Newtonian fluids

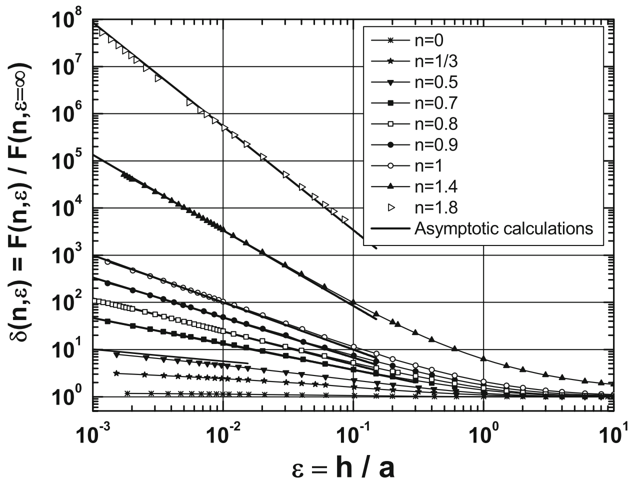

Fig. 5 Influence of the fluidity index on the drag undergone by a sphere approaching a plane in a power-law fluid (*straight lines* asymptotic, *scatter plot* numeric)

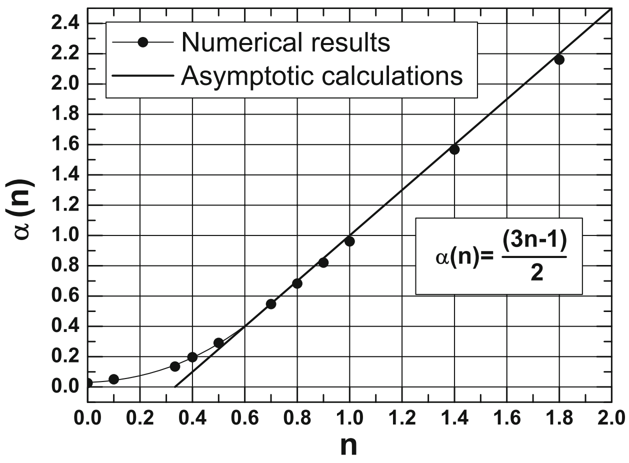

Fig. 6 Comparison of the exponents α (n) of the power-law behaviour obtained numerically and those obtained asymptotically

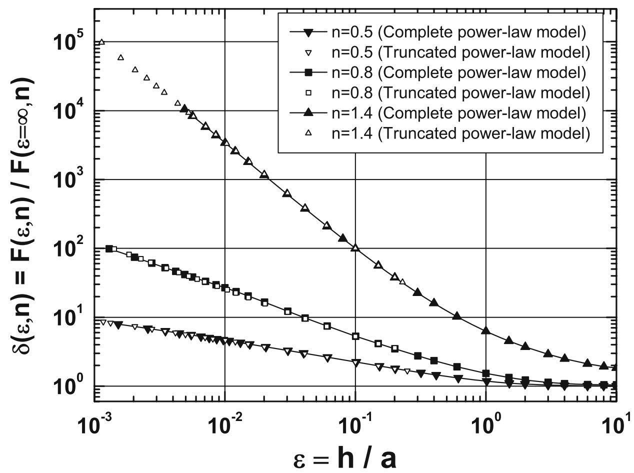

Fig. 7 Comparison of the drag force undergone by a sphere approaching a plane in a complete power-law fluid and that obtained with the truncated Ostwald model

inverse of the maximum of the radial elongational velocity gradient near the stagnation point and consequently the effect of a possible viscoelasticity is negligible (de Gennes 1974; Ambari et al. 1984a), the behaviour of the fluid in this almost Poiseuille-type flow in the lubrication limit (Fig. 2) can be modeled in first approximation by a power-law fluid.

So, in Fig. 5, we give the influence of the index of fluidity n on the drag intensification factor $\delta(n, \varepsilon)$ in the condition of lateral unbounded medium $(k = 10^{-2})$. This perpendicular correction factor corresponds to the normalization of the force undergone by a sphere at a distance $h = \varepsilon a$ from the plane, by the same force in unbounded power-law medium (far from the plane) given by the formulas 5 and 6.

The first remark is the good agreement for $0.6 \leqslant n \leqslant 1.8$, between the numerical results for different confinements and those obtained by the asymptotic approach given by the formula 22 for the lateral unbounded medium. This agreement no longer persists for $n < 0.5$, for which the asymptotic calculation needs to be performed at higher order as we will show below. The second remark concerning Fig. 5 is that the more the fluid is shear thinning, the more the drainage is facilitated (the easier the particles aggregate in dispersions).

Furthermore, Fig. 6, where we give a comparison between the exponent of the power-law behaviour of the drag: $F(n, \varepsilon) \propto \varepsilon^{-\alpha(n)}$ calculated asymptotically: $\alpha(n) = (3n - 1)/2$ and that deduced from the numerical curve (in

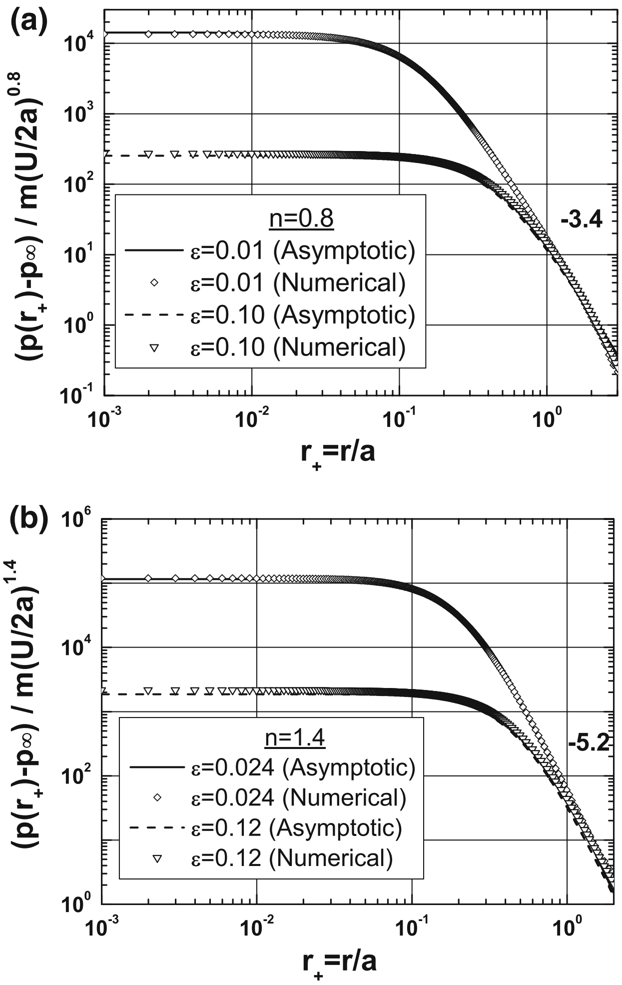

Fig. 8 Comparison of the numerical and the asymptotical pressure distribution in the gap between the sphere and the plane wall for two distances to the wall and different indexes. **a** $n = 0.8$, **b** $n = 1.4$

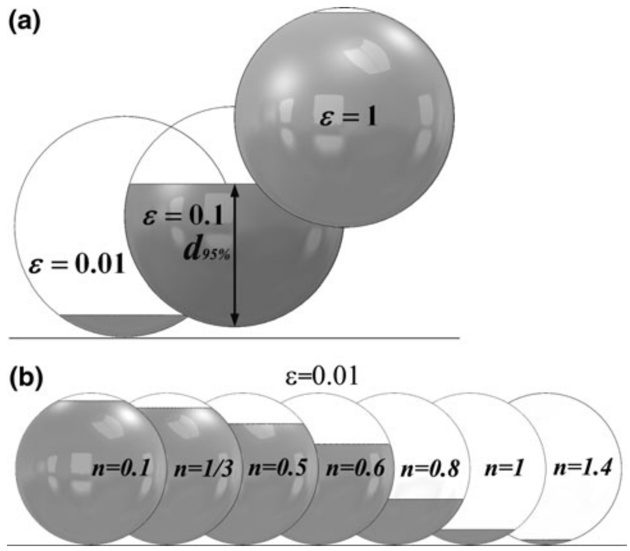

Fig. 9 The dark colored surface corresponds to the area contributing to 95% of the drag undergone by a spherical particle moving towards a plane. **a** Concerns the Newtonian case for three gaps: $\varepsilon = 0.01, 0.1, 1$. **b** Concerns the power-law fluid for a gap $\varepsilon = 0.01$, for three indexes of fluidity $n = 0.8$, 1, 1.4 corresponding respectively to pseudoplastic, Newtonian and dilatant fluids

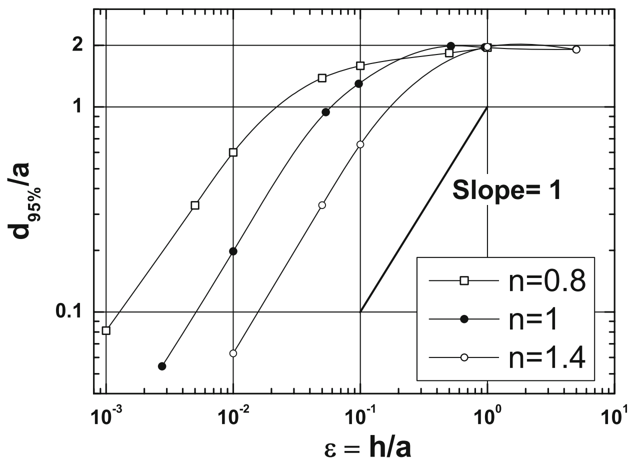

Fig. 10 Linear variation of the liquid film thickness needed to get 95% of the total force which would be undergone by an immersed sphere, versus the normalized distance between the plane and the sphere ε, for different indexes of fluidity n

the lubrication regime) given in Fig. 5, confirms that the asymptotic expression 22 gives accurate results for $n > 0.5$.

As mentioned in Sect. 2, to verify the effect of a truncation at low and high velocity gradients induced by the appearance of both Newtonian plateaus as done by the Carreau–Yasuda model (Carreau et al. 1997), we proceeded to a numerical calculation with an Ostwald truncated model with a variation of the apparent viscosity over three decades (commonly encountered in experiments). The results obtained with this truncated Ostwald model, given in Fig. 7, remain the same as those obtained using a complete Ostwald model and justify its use in this study.

Concerning the radial distribution of the pressure in the gap, in the power-law fluid, we compare successfully in Fig. 8a, b the numerical and asymptotical results (formula 18) for two indexes of fluidity, $n = 0.8$ and $n = 1.4$, corresponding respectively to pseudoplastic and dilatant fluids and two values of the gap. In all cases, the results for Newtonian and power-law fluids confirm the validity of the asymptotic relation 18.

From an experimental point of view, as dynamic surface force apparatus uses a thin liquid film between the spherical tip and the plane, we wanted to know the necessary thickness of this film which allows the use of the formula 22. This height corresponds to that of the part of the sphere, which accounts for 95% of the force undergone by the sphere. Indeed, in Fig. 9a, we show in the Newtonian case that the thickness of this region is all the lower as the gap is small, confirming the lubrication hypothesis. This result confirms that the applicability of the asymptotic law in the lubrication limit depends only on the fore spherical geometry of any particle, as confirmed experimentally for a spherocylinder by Mongruel et al. (2011). However, Fig. 9b corresponding to the power-law fluid (and $\varepsilon = 10^{-2}$) shows that when the fluid is shear thickening $(n = 1.4)$, the height of this zone is significantly reduced. This result confirms the validity of the lubrication hypothesis and explains why the asymptotic calculations are in very good agreement with the numerical calculations for dilatant fluids. On the other hand, in the case where the fluid is shear thinning, this zone increases when the fluidity index n decreases (for a given gap), whence the nonvalidity of the asymptotic calculations at the first order for $0 \leqslant n \leqslant 0.5$, due to the nonvalidity of the lubrication hypothesis. In this range of fluidity indexes n, it would be necessary to pay attention to use a higher thickness of film. Finally, we verified numerically in Fig. 10 that the height $d_{95\%}$ at which 95% of the force is achieved varies linearly in accordance with the gap for its very low values.

5 Conclusion

As encountered in the aggregation process of particles in dispersions or in the dynamic surface force apparatus used in nanorheology, before the contact of the sphere with another sphere or a plane wall, the hydrodynamic force diverges in the lubrication limit. The power-law which describes this behaviour is given numerically for Newtonian fluids. The comparison with the exact solution given by Maude (1961) and Brenner (1961) confirms the validity of the dynamic mesh method used in this geometrically unsteady problem. This successful comparison led us to give a solution to the same problem, under the quasi-steady state assumption, to the Ostwald–de Waele fluid. In this case, the power-law of the divergence of the force and the distribution of the pressure in the gap are given asymptotically in the lubrication limit and compared successfully to the numerical results for $0.5 < n \leqslant 1.8$. These asymptotic and numerical solutions show clearly that, when the sphere moves towards the plane, the more the fluid shows a shear thinning behaviour, the lower the increase in the perpendicular correction factor. Then the aggregation of particles is facilitated in shear thinning fluids, with regard to the Newtonian case. The inverse effect occurs in the case of the shear thickening fluid. This new result can also find an application in the study of aggregation of dispersions and surface force apparatus for nanorheology in power-law fluids.

6 Asymptotic results

At very low Reynolds numbers, when the sphere approaches the wall at constant velocity in unbounded lateral medium, in the limit of the lubrication regime $\varepsilon \ll 1$ (see Fig. 2), the drag force is assumed to be controlled principally by the drainage process of the liquid film located in the gap remaining between the sphere and the plane. So the drag force submitted by a sphere can be calculated from the pressure force induced by the radially ejected flow. This drainage taking place in this gap, for power-law fluid, is reduced to a "radial power-law Poiseuille flow" as shown in Fig. 2 where $a\varepsilon_1(\theta) = a\varepsilon + a(1 - \cos\theta)$ and $a\varepsilon$ is the minimum gap between the sphere and the plane. In fact, starting from the following reduced momentum and continuity equations in the lubrication limit:

$$\frac{\partial}{\partial z}\left[m\left|\frac{\partial u_r(r,z)}{\partial z}\right|^n\right] = \frac{\partial p}{\partial r} \tag{13}$$

$$\frac{1}{r}\frac{\partial}{\partial r}[ru_r(r,z)] + \frac{\partial u_z}{\partial z} = 0 \tag{14}$$

By successive integrations of Eq. (13) and using the boundary conditions, one can obtain:

$$u_r(r,z) = \frac{n}{n+1}\left[-\frac{1}{m}\frac{\partial p}{\partial r}\right]^{\frac{1}{n}} \times \left[\left|z - \frac{a\varepsilon_1(\theta)}{2}\right|^{1+\frac{1}{n}} - \left(\frac{a\varepsilon_1(\theta)}{2}\right)^{1+\frac{1}{n}}\right] \tag{15}$$

So by integrating the continuity Eq. (14) over the gap and taking into account that the velocity of the particle $u_z(r,z)|_{sphere} = -U$

$$\frac{1}{r}\int_{-\frac{a\varepsilon_1}{2}}^{+\frac{a\varepsilon_1}{2}} \frac{\partial}{\partial r}[ru_r(r,z)]\mathrm{d}z + \int_{-\frac{a\varepsilon_1}{2}}^{+\frac{a\varepsilon_1}{2}} \frac{\partial u_z(r,z)}{\partial z}\mathrm{d}z = 0 \tag{16}$$

where $u_r(r, z)$ is given by Eq. (15), $u_z(r, z = +a\varepsilon_1/2) = -U$ on the sphere and $u_z(r, z = -a\varepsilon_1/2) = 0$ on the plane, we obtain the radial pressure distribution along the gap which is given by:

$$p(r_+) - p_\infty = -mU^n\left(\frac{2n+1}{n}\right)^n \times \left(\frac{2^{\frac{n+1}{2}}}{a^n\varepsilon^{\frac{3n+1}{2}}}\right)\int_{+\infty}^{\frac{1}{2\varepsilon}r_+^2} \frac{Y^{\frac{n-1}{2}}}{[Y+1]^{2n+1}}\mathrm{d}Y \tag{17}$$

where $Y = \frac{1}{2\varepsilon}r_+^2$ and $r_+ = r/a$ is the normalized radial distance from the stagnation point. Using the "MATHE MATICA" code, the normalized radial distribution of the pressure is given by:

$$\frac{p(r_+) - p_\infty}{m\left(\frac{U}{2a}\right)^n} = \left(\frac{2n+1}{n}\right)^n \left(\frac{2^{2+3n}}{1+3n}\right) \times {}_2F_1\left(1+2n, \frac{1}{2}(1+3n); \frac{3}{2}(1+n); -2\frac{\varepsilon}{r_+^2}\right)(r_+)^{-(1+3n)} \quad (18)$$

where p_∞ is the pressure in the farfield and ${}_2F_1$ is the hypergeometric function. Note that this result is similar to that obtained from Rodin's formula 7 and recalled in Sect. 3. Nevertheless, his expression of the pressure at the axis (his formula 23, Rodin (1996)) is not accurate and must be replaced by:

$$\hat{p}(0) = \frac{\alpha^{\frac{1+3n}{2}} \pi \Gamma\left(\frac{3+3n}{2}\right)}{\cos\left(\frac{n\pi}{2}\right) \Gamma\left(\frac{1-n}{2}\right) \Gamma(1+2n)} \quad (19)$$

Then the pressure in the axis is given by, for $n \neq 1$:

$$\frac{p(r_+ = 0) - p_\infty}{m\left(\frac{U}{2a}\right)^n} = \frac{\pi 2^{\frac{3+3n}{2}}}{1+3n}\left(\frac{2n+1}{n}\right)^n \times \frac{\Gamma\left(\frac{3+3n}{2}\right)}{\cos\left(\frac{n\pi}{2}\right)\Gamma\left(\frac{1-n}{2}\right)\Gamma(1+2n)} \frac{1}{\varepsilon^{\frac{1+3n}{2}}} \quad (20)$$

For $n = 1$, the limit of the pressure in this Newtonian case is equal to:

$$\frac{p(r_+ = 0) - p_\infty}{\eta\left(\frac{U}{2a}\right)} = \frac{6}{\varepsilon^2} \quad (21)$$

which is the same value that can be obtained from Eq. (8). In this approach, the correction factor $\delta(n, k = 0, \varepsilon)$ of the drag undergone by a sphere can be calculated by integrating the pressure given by formula 18 over the frontal surface of the sphere in the lubrication limit:

$$\delta(n, k = 0, \varepsilon) = \frac{F(n, k = 0, \varepsilon)}{F(n, k = 0, \varepsilon \to \infty)} = \frac{2^{\frac{3n+3}{2}}}{3(9n^2 - 1)}\left(\frac{2n+1}{n}\right)^n \times \frac{\Gamma\left(\frac{3n+3}{2}\right)\Gamma\left(\frac{n+3}{2}\right)}{\Gamma(2n+1)} \frac{1}{\varepsilon^{\frac{3n-1}{2}}} \quad (22)$$

This formula, which is valid mathematically only for $n > 1/3$, reduces to the formula 9 deduced from Rodin's result in the limit of $\beta = \infty$ (Sect. 3). In fact, to verify the equivalency between both formulae, let us recall that (Gradshteyn and Ryzhik 1983):

$$\frac{4}{9n^2 - 1} \frac{\Gamma\left(\frac{3+n}{2}\right)\Gamma\left(\frac{3n+3}{2}\right)}{\Gamma(2n+1)} = \beta\left(\frac{3+n}{2}, \frac{3n-1}{2}\right) \quad (23)$$

In addition, for $n = 1/3$, the pressure field is given by:

$$\frac{p(r_+) - p_\infty}{m\left(\frac{U}{2a}\right)^{\frac{1}{3}}} = 3 \times 5^{\frac{1}{3}} \frac{1}{\varepsilon}\left[1 - \frac{1}{\left(\frac{2\varepsilon}{r_+^2} + 1\right)^{\frac{2}{3}}}\right] \quad (24)$$

But the calculation of the force from integration of Eq. (24) is divergent. Furthermore, the comparison of our numerical results with the asymptotic solution given by Rodin (formulae 10 and 11) proves that these last results are no longer valid. However, due to the dependence of the drag on the flow in the upper half of the sphere, shown by the numerical results in Fig. 9b, it is not possible to give the required higher order to our asymptotic calculation. In fact, the numerical calculation turns out to be necessary as also confirmed by Fig. 6. So all the asymptotic results given by Rodin (1996) and us are no longer applicable for $n < 0.6$. Therefore,

$$F(n, \varepsilon) = -m a^{(n+3)/2} \pi 2^{(n+7)/2} \frac{\delta(n, \varepsilon \to \infty)}{9n^2 - 1} \times \left(\frac{2n+1}{n}\right)^n \frac{\Gamma\left(\frac{3n+3}{2}\right)\Gamma\left(\frac{n+3}{2}\right)}{\Gamma(2n+1)} \frac{1}{h^{(3n-1)/2}} \left|\frac{\mathrm{d}h}{\mathrm{d}t}\right|^n \quad (25)$$

which reduces, in the Newtonian case, to the classical relation (Vinogradova 1995):

$$F(n = 1, \varepsilon) = -\frac{6\pi\eta a^2}{h}\left|\frac{\mathrm{d}h}{\mathrm{d}t}\right| \quad (26)$$

References

Ambari A (1979) Ecoulement des solutions aqueuses diluées de polyoxyéthylène dans une fente rectangulaire très allongée en mince paroi. C R Acad Sci Paris Série B 289:511–530

Ambari A, Deslouis C, Tribollet B (1984a) Coil-stretch transition of macromolecules in laminar flow around a small cylinder. Chem Eng Commun 29:63–78

Ambari A, Gauthier-Manuel B, Guyon E (1984b) Wall effects on a sphere translating at constant velocity. J Fluid Mech 149: 235–253

Ardekani AM, Joseph DD, Dunn-Rankin D, Rangel RH (2009) Particle-wall collision in a viscoelastic fluid. J Fluid Mech 633:475–483

Bhushan B (2010) Springer handbook of nanotechnology, 3rd edn. Springer

Brenner H (1961) The slow motion of a sphere through a viscous fluid towards a plane surface. Chem Eng Sci 16:242–251

Carreau PJ, Kee DCRD, Chhabra RP (1997) Rheology of polymeric systems: principles and applications. Hanser Gardner Publications

Chan DY, Horn RG (1985) The drainage of thin liquid films between solid surfaces. J Chem Phys 83:5311–5324

Cottin-Bizonne C (2003) Hydrodynamique à l'interface solide-liquide: étude par mesures de forces de surfaces et simulations de dynamique moléculaire. Ph.D. thesis, Université Claude Bernard Lyon I

Cox R, Brenner H (1967) The slow motion of a sphere through a viscous fluid towards a plane surface. part ii. small gaps widths, including inertial effects. Chem Eng Sci 22:1753–1777

Cox RG (1974) The motion of suspended particles almost in contact. Int J Multiphase Flow 1:343–371

de Gennes PG (1974) Coil-stretch transition of dilute flexible polymers under ultrahigh velocity gradients. J Chem Phys 60:5030–5042

Despeyroux A, Ambari A, Ben Richou A (2011) The hydrodynamic interaction effects on the settling and the transportation of a sphere in power-law fluids. Phys Fluids (under revision)

Gradshteyn IS, Ryzhik IM (1983) Tables of integrals, series, and products, 4th edn. Academic Press, Inc., Orlando

Guala M, Stocchino A (2007) Large-scale flow structures in particle–wall collision at low Deborah numbers. Eur J Mech B Fluids 26:511–530

Hinch EJ (1974) Mechanical models of dilute polymer solutions for strong flows with large polymer deformations. In: Colloque Intern. CNRS: polymère et lubrification, Brest, pp 241–247

Hocking LM (1973) The effect of slip on the motion of a sphere close to a wall and of two adjacent spheres. J Eng Math 7:207–221

Horn RG, Vinogradova OI, Mackay ME, Phan-Thien N (2000) Hydrodynamic slippage inferred from thin film drainage measurements in a solution of nonadsorbing polymer. J Chem Phys 112:6424–6433

Luengo G, Schmitt FJ, Hill R, Israelachvili J (1997) Thin film rheology and the tribology of confined polymer melts: contrasts with bulk properties. Macromolecules 30:2482–2494

Marston JO, Yong W, Thoroddsen ST (2010) Direct verification of the lubrication force on a sphere travelling through a viscous film upon approach to a solid wall. J Fluid Mech 655:515–526

Maude AD (1961) End effects in a falling-sphere viscometer. Br J Appl Phys 12:293–295

Mongruel A, Lecoq N, Wajnryb E, Cichocki B, Feuillebois F (2011) Motion of a sphero-cylindrical particle in a viscous fluid in confined geometry. Eur J Mech B Fluids 30:405–408

Pasol L, Chaoui M, Yahiaoui S, Feuillebois F (2005) Analytical solutions for a spherical particle near a wall in axisymmetrical polynomial creeping flows. Phys Fluids 17:073,602

Potanin AA, Ur'ev NB, Muller VM (1988) Kolloidn Zh 50:493

Restagno F, Crassous J, Charlaix E, Cottin-Bizonne C, Monchanin M (2002) A new surface apparatus for nanorheology. Rev Sci Instrum 73:2292–2297

Rodin GJ (1996) Squeeze film between two spheres in a power-law fluid. J Non Newtonian Fluid Mech 63:141–152

Stimson M, Jeffery GB (1926) The motion of two spheres in a viscous fluid. Proc R Soc Lond Ser A 111:110–116

Stocchino A, Guala M (2005) Particle-wall collision in shear thinning fluids. Exp Fluids 38:476–484

Vinogradova OI (1995) Drainage of a thin liquid film confined between hydrophobic surfaces. Langmuir 11:2213–2220