doi: 10.14419/ijet.v4i2.4570

Transverse vibration analysis of Euler-Bernoulli beam carrying point masse submerged in fluid media

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Abstract

In the present paper, an analytical method is developed to investigate the effects of added mass on natural frequencies and mode shapes of Euler-Bernoulli beams carrying concentrated masse at arbitrary position submerged in a fluid media. A fixed-fixed beams carrying concentrated masse vibrating in a fluid is modeled using the Bernoulli-Euler equation for the beams and the acoustic equation for the fluid. The symbolic software Mathematica is used in order to find the coupled vibration frequencies of a beams with two portions. The frequency equation is deduced and analytically solved. The finite element method using Comsol Multiphysics software results are compared with present method for validation and an acceptable match between them were obtained. In the eigenanalysis, the frequency equation is generated by satisfying all boundary conditions. It is shown that the present formulation is an appropriate and new approach to tackle the problem with good accuracy.

Keywords: *Euler-Bernoulli beam; Fluid-structure interaction; Finite element method; Frequency equation; Inertial coupling.*

1. Introduction

Systems of a beam or rod carrying masses are frequently used as design models in engineering. The operation of the machine may introduce severe dynamic stresses on the beam. It is important, then, to know the natural frequencies of the coupled beam-mass and beam-mass-fluid system, in order to obtain a proper design of the structural components. Numerous papers have been published on the vibration analysis of beams carrying concentrated masses at arbitrary positions [1, 2, 3, 4, 5]. The problem of a vibrating simply supported beam carrying a concentrated mass at its center was solved analytically by [6]. The Laplace transformation technique to formulate the frequency equation for beams carrying intermediate concentrated masses have been used in [7]. In [8] Rayleigh-Ritz method was used to study continuous beams subjected to axial forces and carrying concentrated masses. Laura [9] studied the cantilever beam carrying a lumped mass at the top in order to obtain analytical solution by introducing the mass in the boundary conditions. Other studies of the influence of these factors on natural frequencies of beams and rods are given in [10, 11, 12, 13, 14, 15].

In most of the studies mentioned above the obtaining the natural frequencies of beams carrying point masses

is performed in vacuum. This paper presents an aspect of fluid-structure interaction in order to investigate the influence of the added mass due to fluid on the natural frequencies of a beams carrying point masses submerged in fluid (water). Fluid-structure interaction problems since long have attracted the attention of engineers and applied mathematics. The most important applications of this theory, probably, structural acoustics [16], vibrations of fluid-conveying pipes [17] and biomechanics. In fact many engineering fields have related interest, including naval architecture, offshore structures, hydrodynamics, dam-reservoir systems under seismic, noise control and vibration isolation, as well as nuclear reactor plants. As these problems are rather complicated, some simplifications are typically adopted to facilitate their solving. In particular, it's quite typical to ignore viscosity effects (especially in structural acoustics) or to use local theories of interaction, such as, the one referred to as thin layer or plane wave approximation. Therefore vibrations of fluid-coupled beams have received a great attention due to their importance in various engineering applications. As a consequence of the fluid-structure interaction, there are coupled dynamical equations to be solved simultaneously. A number of papers has been published in recent years investigating the added mass effects of the interacting fluids in structures. Xing [18] examined the dynamical behaviour of a flexible beam-water interaction system and showed that in the water domain, the natural frequencies of the coupled dynamic system are lower than those of the flexible dry beam, indicating that the influence of water on the beam has the effect of an additional mass.

The present manuscript deals with the coupling effect of liquid on the free vibration characteristics of a Euler-Bernoulli beams carrying point masse submerged in water contained in a rigid rectangular container. The originality of this study is to investigate the effects of both pointed mass and fluid interaction on the dynamics of beams under the fixed-fixed boundary condition at the beam edge. The governing equations describing the behavior of the system are analyzed using the separation of variables method and their solutions presented. The eigenvalue equation of the natural vibration of the beam-water system is derived and exact solutions are obtained. Natural frequencies and modes shapes of the beam-masse-water are investigated. The theoretical results from the present formulation are compared to those from a finite element analysis using a commercial finite element analysis. Calculations show that, in the water domain, the concentrated mass become less influential on the natural vibration behavior of the coupled system. Finally, one investigates how the free surface affects the frequencies. Finding a general analytical solution for Euler-Bernoulli beams carrying point masse submerged in fluid remains a challenge.

2. Governing equations of fluid domain

First of all, the fluid motion must be defined in order to find the coupled natural frequencies and wet mode shapes of the beams carrying point masses in contact with the fluid. To account for the effect of the added mass in the beam response, the acoustic mode shapes and natural frequencies are first computed for the rectangular domain fluid assuming rigid and soft boundary conditions. The fluid is assumed non-viscous and isotropic which satisfies the acoustic wave equation. Since no source is considered in the acoustic domain, the Helmholtz equation, which describes a harmonic wave equation propagating in medium while neglecting dissipation, is represented as

$$\nabla^2 p + \frac{\omega^2}{c^2} p = 0 \tag{1}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian operator in a cartesian coordinate system. p is the acoustic pressure in a fluid medium, ω is the angular frequency and speed of sound c. The Eq. (1) indicates the distribution of the acoustic pressure p in space and can be solved if the boundary condition is knew well. The natural frequencies of the acoustical system are obtained by assuming that the boundaries of the enclosure are hard except the boundary Γ_p (zero free surface wave disturbance), hence the pressure gradients on the boundaries Γ_1, Γ_2 and Γ_3 are set to zero :

$$\frac{\partial p}{\partial x}(0, y) = 0 \ , \quad \frac{\partial p}{\partial x}(L, y) = 0 \ , \quad \frac{\partial p}{\partial y}(x, 0) = 0 \ , \quad p(x, H) = 0 \tag{2}$$

The solution of (1) with boundary conditions introduced in (2) is given as

$$p(x, y) = P \cos(\alpha_n x) \cos(\alpha_q y) \ , \ n, q = 0, 1, 2, ... \tag{3}$$

where $\alpha_n = n\pi/L$, $\alpha_q = (2q+1)\pi/2H$ and P is unknown coefficient which will be determined later by imposing the appropriate boundary conditions.

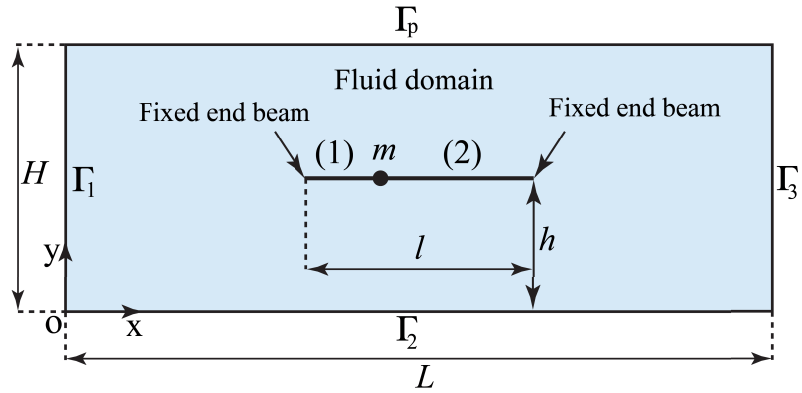


Figure 1: Sketch of beam composed of 2 uniform beam segments (denoted by (1) and (2)) carrying a point masse (denoted m) submerged in a liquid-filled rectangular geometry.

3. Theoretical modal analysis of beams

Modal analysis of an elastic submerged structures is needed in every modern construction and have wide engineering application especially in ocean engineering. In this study, modal analysis is important to predict the dynamic behavior of the submerged beams. It is well known that the natural frequencies of the submerged elastic structures are different from those in vacuum. The effect of fluid forces on the submerged beams is represented as added mass, which decreases the natural frequencies of the submerged beams from those which would be measured in the vacuum. This decrease in the natural frequencies is caused by the increase of the kinetic energy of the fluid-beams system without a corresponding increase in strain energy.
The model allows to analyze the influence of the added masse on the dynamic behavior of beams with two classic boundary conditions. The Euler-Bernoulli beams carrying point masse is submerged inside rectangular fluid domain where the lower liquid region is represented by h. The interaction between the fluid and the Euler-Bernoulli beams carrying point mass is taken into account to calculate the natural frequencies and modes shapes of the coupled system. The dynamics of each beam portion are treated separately. It is assumed that the beam has aligned neutral axis. The equation of motion for the transverse deflection of a uniform elastic beam ignoring shear deformation and rotary inertia effects can be written in the form

$$EI\frac{\partial^4 v_i(x,t)}{\partial x^4} + \rho S\frac{\partial^2 v_i(x,t)}{\partial t^2} = p(x,h,t)\ , \quad i = 1,2 \tag{4}$$

where $v_i(x,t)$ is the lateral deflection at distance x along the length of the beam, EI, ρ and S are the flexural rigidity, the mass per unit volume and the cross-sectional area of the beams. Assuming time harmonic motion at angular frequency of the form $\exp(j\omega t)$, but neglecting it in the subsequent expressions for clarity, the general solutions of the ordinary differential Eq. (4) for the beams system, as shown in Figure 1, can be written in different segments in terms of evanescent and propagating waves as

$$v_i(x) = A_i\phi(x) + B_i\psi(x) + C_i\chi(x) + D_i\varphi(x) + \frac{P\cos(\alpha_m h)}{EI\,(\alpha_n^4 - k^4)}\cos(\alpha_n x) \tag{5}$$

where the trigonometric functions

$$\phi(x) = \sin(kx)\ ,\ \psi(x) = \cos(kx)$$

represent the propagating waves, and the hyperbolic functions

$$\chi(x) = \sinh(kx)\ ,\ \varphi(x) = \cosh(kx)$$

represent the evanescent waves. A_i is a wave propagating to the left, B_i is a wave propagating to the right, C_i is an evanescent wave decaying to the left and D_i is an evanescent wave decaying to the right; k is the flexural wavenumber and is given by

$$k = \sqrt{\omega}\left(\frac{\rho S}{EI}\right)^{\frac{1}{4}}$$

The nine constants P, A_i, B_i, C_i and D_i $(i = 1, 2)$ can be found by imposing the following boundary conditions. Eq. (5) is the general solution for the vibration modes of beams submerged in fully liquid. In the case of a coupled system, the effect of sound field on the flexible beams carrying concentrated masse must be considered. On the fluid-beams interface, the normal acceleration must be continuous. Therefore, the pressure in the fluid and the deflection of a beams v_i satisfy the relation :

$$\frac{\partial p(x,h)}{\partial y} = \rho_f \omega^2 v_i(x) \tag{6}$$

where ρ_f is the density of the fluid. Inserting Eq. (3) into Eq. (6) and substituting the outcome into Eq. (5) and after some manipulations, leading to the following equation :

$$A_i\phi(x) + B_i\psi(x) + C_i\chi(x) + D_i\varphi(x) = -P\left[\frac{\alpha_m \sin(\alpha_m h)}{\rho_f\omega^2} + \frac{\cos(\alpha_m h)}{EI\left(\alpha_n^4 - k^4\right)}\right]\cos(\alpha_n x) \tag{7}$$

Making use of the orthogonality of trigonometric functions, both sides of the Eq. (7) are multiplied by $\cos(\alpha_j x)$ and then integrated over $0 < x < L$ to yield the following equation :

$$A_iU + B_iX + C_iY + D_iZ = -\frac{PL}{2}\left[\frac{\alpha_m \sin(\alpha_m h)}{\rho_f\omega^2} + \frac{\cos(\alpha_m h)}{EI\left(\alpha_n^4 - k^4\right)}\right] \tag{8}$$

where the coefficients U, X, Y and Z are defined as follows :

$$U = \int_0^L \phi(x)\cos(\alpha_n x)dx \ , \quad X = \int_0^L \psi(x)\cos(\alpha_n x)dx$$
$$Y = \int_0^L \chi(x)\cos(\alpha_n x)dx \ , \quad Z = \int_0^L \varphi(x)\cos(\alpha_n x)dx$$

Now the expression of the lateral movement of the beams, fully in contact with the fluid $v_i(x)$ can be formulated using Eq. (5) and taking into account Eq. (8) as :

$$\begin{aligned} v_i(x) = {} & A_i\left[\phi(x) - \beta U\cos(\alpha_n x)\right] + B_i\left[\psi(x) - \beta X\cos(\alpha_n x)\right] + C_i\left[\chi(x) - \beta Y\cos(\alpha_n x)\right] \\ & + D_i\left[\varphi(x) - \beta Z\cos(\alpha_n x)\right] \end{aligned} \tag{9}$$

where

$$\beta = \frac{2}{L\left[\dfrac{EI\left(\alpha_n^4 - k^4\right)\alpha_m \tan(\alpha_m h)}{\rho_f\omega^2} + 1\right]}$$

From equation (9), it can be seen that, when $\beta \to 0$, the coupled system beams-mass-water is reduced to the case of beams motion in vacuo. Now, to derive the frequency equation of beams submerged in fully fluid, one assumes that on the interface between each portions of the beam at the position l_1, the deflection, the rotation angle, the internal shear force and bending moment of the beam must be continuous. This is satisfied when

$$\begin{aligned} & v_1(l_1) = v_2(l_1) \\ & v_1'(l_1) = v_2'(l_1) \\ & v_1''(l_1) = v_2''(l_1) \\ & EIv_1'''(l_1) + m\omega^2 v_1(l_1) = EIv_2'''(l_1) \end{aligned} \tag{10}$$

where primes denote differentiation with respect to the position variable x. To complete the formulation of the boundary-value problem, the four boundary conditions (left and right beam ends) for the beam ends considered in this work are specified as follows :

$$v_1(0) = v_1'(0) = v_2(l) = v_2'(l) = 0 \tag{11}$$

The boundary conditions (Eqs. 10 and 11) can be evaluated, giving rise to a linear homogeneous system of eight equations :

$$\mathbf{Nx} = \mathbf{0} \ , \ \mathbf{x} = \left[A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2\right]^T \tag{12}$$

This system can have non trivial solutions only if the determinant of the matrix $\mathbf{N}$ is zero, leading to the frequency equation that takes the following form $\mathbf{N} = \left[\mathbf{N}_{11}, \mathbf{N}_{21}, \mathbf{N}_{31}, \mathbf{N}_{41}, \mathbf{N}_{51}, \mathbf{N}_{61}, \mathbf{N}_{71}, \mathbf{N}_{81}\right]^T$ where the matrix $\mathbf{N}$ is written down explicitly in the Appendix.

4. Results and discussion

In this paper, the results obtained by the present analytical solution for the beams carrying point masse coupled with fluid are compared with those acquired by the finite element method (FEM) using Comsol Multiphysics FEM Simulation Software to show the applicability, reliability and effectiveness of the presented formulation. The mechanical and geometrical properties of the beams used in numerical computation are :
Young's modulus $E = 13 \cdot 10^8$ [Pa], material density $\rho = 2000\,[\mathrm{kg} \cdot \mathrm{m}^{-3}]$, Poisson's ratio $\nu = 0.3$, length $l = 0.6$ [m], width $a = 0.006$ [m] and thickness $e = 0.006$ [m]. The fluid density is taken to be $\rho_f = 1000\,[\mathrm{kg} \cdot \mathrm{m}^{-3}]$ and the speed of sound in a fluid $c = 1500\,[\mathrm{m} \cdot \mathrm{s}^{-1}]$. The modeling was set up by placing the beams in a rectangular reservoir with dimensions $L = 3$ [m] and $H = 1$ [m] and lower fluid region $h = 0.5$ [m]. The mass of attached masse $m = 0.04$ [kg]. In Tables 1 and 2, natural frequencies of beams in (Hz) unit carrying or not point masse obtained by the present theory are presented and compared with numerical data. As can be seen, there is a very good agreement between the present results and those of FEM and the relative difference ((FEM-Present)/Present) is $\leqslant 0.1\%$. This shows that the algorithm implemented in Comsol Multiphysics [19] software for numerical computation is highly reliable and accurate. Is the attention to use the numerical formulation in future for more general geometries and other kind of fluid-structure interactions such as those arising in aeroelasticity for example.

Table 1: The first six natural frequencies in "vacuo" of the beams with point mass neglected and considered.

Modes	Neglected		Considered				Diference
			at $l/3$		at $l/2$		
	Present	FEM	Present	FEM	Present	FEM	(%)
1	13.81	13.81	8.57	8.57	7.50	7.50	0
2	38.07	38.07	29.10	29.10	38.07	38.07	0
3	74.64	74.64	73.37	73.37	59.36	59.36	0
4	123.38	123.38	110.07	110.07	123.38	123.38	0
5	184.31	184.31	162.20	162.20	156.94	156.94	0
6	257.43	257.43	255.06	255.06	257.43	257.43	0

Table 2: The first six natural frequencies in "water" of the beams with point mass neglected and considered in the case of zero free surface wave disturbance.

Modes	Neglected		Considered				Diference
			at $l_1 = l/3$		at $l_1 = l/2$		
	Present	FEM	Present	FEM	Present	FEM	(%)
1	1.76	1.76	1.74	1.74	1.73	1.73	0
2	8.34	8.34	8.00	8.00	8.34	8.34	0
3	19.82	19.82	19.82	19.82	17.99	17.99	0
4	38.49	38.49	35.71	35.71	38.49	38.49	0
5	63.84	63.84	58.18	58.18	58.07	58.07	0
6	97.68	97.68	97.34	97.34	97.68	97.68	0

This present work, the effects of concentrated mass and add mass on the coupled natural frequencies of beams are presented with the present method.
First, one investigates how the dense fluid (added mass) affects the natural frequencies. Due to the practical applications, the natural frequencies of beams coupled with fluid are listed in Table 2. It is evident from Table 2 that when the beams is submerged fluid, the natural frequencies of the beams takes lower values. Also, tables 1 and 2 show that when the concentrated mass is placed at $x = l/2$ (symmetric problem), only the symmetric modes (odd modes) are affected.

Secondly, in order to see the influence of concentrated mass m on the vibration characteristics of the beams, the first six natural frequencies of beams are listed in Table 1. It is seen that the concentrated mass decreases the natural frequencies in vacuo (higher 20%). In water, it is interesting to note that the concentrated mass become less influential on the natural frequencies of the beams (Table 2). Figures 2-7 show also that the concentrated mass modifies the modal shapes in vacuum. The presence of the fluid has low influence on the modal shapes because the added mass is diagonal (Figures 8-13). The coupling is light in such a case.

Thirdly, one investigates how the free surface affects the frequencies. The above mentioned discussions are obtained in the case of zero free surface wave disturbance ($p = 0$ at Γ_p in Figure 1). In order to investigate the influence of the free surface on the natural frequencies of beams, the free surface wave disturbance Γ_p is governed by the equation

$$\frac{\partial p}{\partial y} = -\frac{1}{g}\frac{\partial^2 p}{\partial t^2} \tag{13}$$

and $g = 9.81$ $[\mathrm{m} \cdot \mathrm{s}^{-2}]$ is the acceleration due to gravity. Figures 14-19 shows the natural frequencies and mode shapes of the submerged beams without point mass, varying with the free surface. The free surface decreases significantly the natural frequency. This behavior is also found for submerged beams with point mass.

The study also shows that the concentrated mass do not affect the coupled natural frequencies in the case of free surface wave disturbance. Comparing different boundary conditions, it is observed that added mass factor has a greater effect with free surface on the coupled natural frequencies. Therefore, the effect of free surface is very significant.

(a) $f = 13.81$ Hz (b) $f = 8.57$ Hz (c) $f = 7.50$ Hz

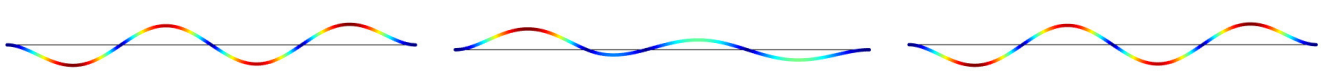
Figure 2: The first natural frequency and mode shape of beams in "vacuo": the colors pertain to the displacement field of beams. (a) without point mass. (b) with point mass at $l/3$. (c) with point mass at $l/2$.

(a) $f = 38.07$ Hz (b) $f = 29.10$ Hz (c) $f = 38.07$ Hz

Figure 3: The second natural frequency and mode shape of beams in "vacuo": the colors pertain to the displacement field of beams. (a) without point mass. (b) with point mass at $l/3$. (c) with point mass at $l/2$.

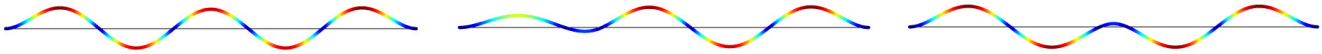
(a) $f = 74.64$ Hz (b) $f = 73.37$ Hz (c) $f = 59.36$ Hz

Figure 4: The third natural frequency and mode shape of beams in "vacuo": the colors pertain to the displacement field of beams. (a) without point mass. (b) with point mass at $l/3$. (c) with point mass at $l/2$.



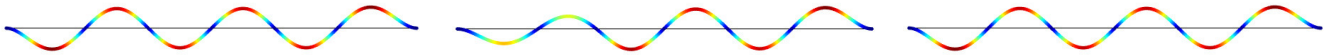
(a) $f = 123.38$ Hz (b) $f = 110.07$ Hz (c) $f = 123.38$ Hz

Figure 5: The fourth natural frequency and mode shape of beams in "vacuo": the colors pertain to the displacement field of beams. (a) without point mass. (b) with point mass at $l/3$. (c) with point mass at $l/2$.



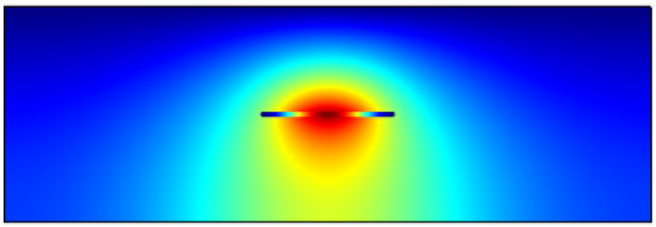
(a) $f = 184.31$ Hz (b) $f = 162.20$ Hz (c) $f = 156.94$ Hz

Figure 6: The fifth natural frequency and mode shape of beams in "vacuo": the colors pertain to the displacement field of beams. (a) without point mass. (b) with point mass at $l/3$. (c) with point mass at $l/2$.



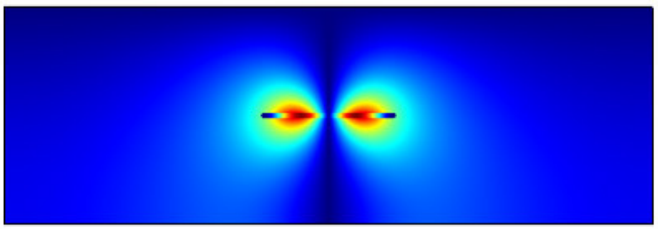
(a) $f = 257.43$ Hz (b) $f = 255.06$ Hz (c) $f = 257.43$ Hz

Figure 7: The sixth natural frequency and mode shape of beams in "vacuo": the colors pertain to the displacement field of beams. (a) without point mass. (b) with point mass at $l/3$. (c) with point mass at $l/2$.



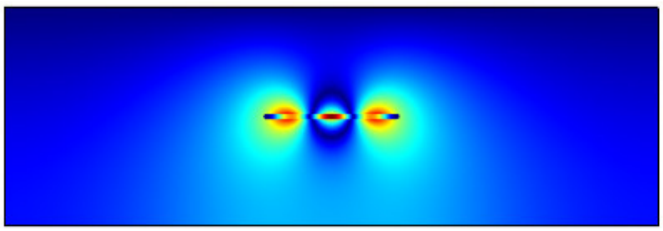
(a) $f = 1.76$ Hz

Figure 8: The first natural frequency and mode shape of beams in water in the case of zero free surface wave disturbance. The colors pertain to the displacement field of beams and to the absolute acoustic pressure.



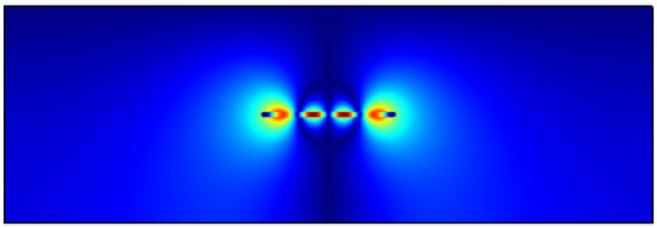
(a) $f = 8.34$ Hz

Figure 9: The second natural frequency and mode shape of beams in water in the case of zero free surface wave disturbance. The colors pertain to the displacement field of beams and to the absolute acoustic pressure.



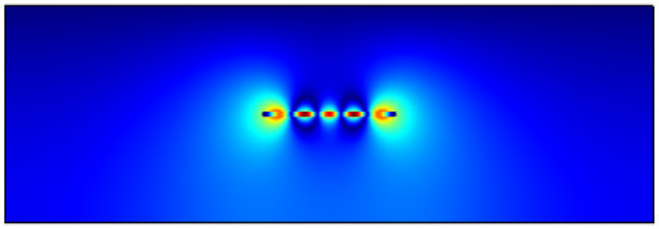
(a) $f = 19.82$ Hz

Figure 10: The third natural frequency and mode shape of beams in water in the case of zero free surface wave disturbance. The colors pertain to the displacement field of beams and to the absolute acoustic pressure.



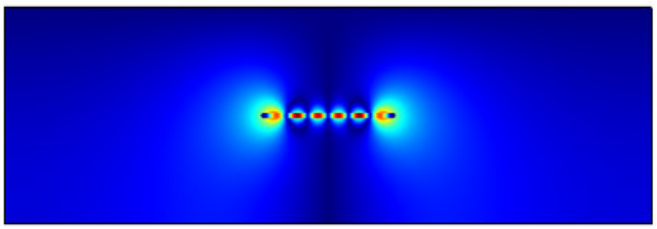
(a) $f = 38.49$ Hz

Figure 11: The fourth natural frequency and mode shape of beams in water in the case of zero free surface wave disturbance. The colors pertain to the displacement field of beams and to the absolute acoustic pressure.



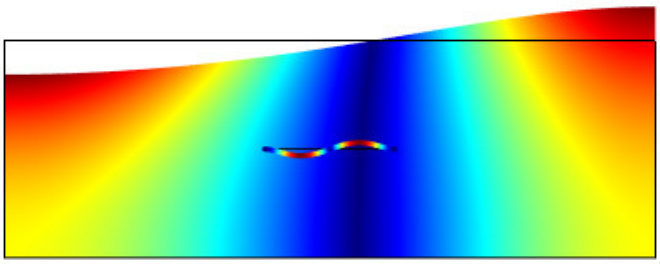
(a) $f = 63.84$ Hz

Figure 12: The fifth natural frequency and mode shape of beams in water in the case of zero free surface wave disturbance. The colors pertain to the displacement field of beams and to the absolute acoustic pressure.



(a) $f = 97.68$ Hz

Figure 13: The sixth natural frequency and mode shape of beams in water in the case of zero free surface wave disturbance. The colors pertain to the displacement field of beams and to the absolute acoustic pressure.

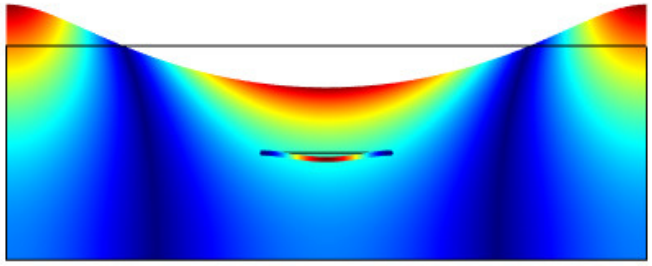


(a) $f = 0.014$ Hz

Figure 14: The first coupled natural frequency and mode shape of beams in water in the case of free surface wave disturbance. The colors pertain to the displacement field of beams and to the absolute acoustic pressure.

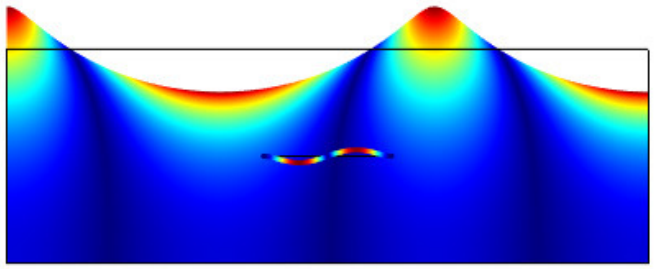
5. Conclusion

This paper presents an analytical solution for the dynamic system of Euler-Bernoulli beams carrying point mass submerged in fluid media. Separation variable technique and orthogonality of trigonometric functions are used



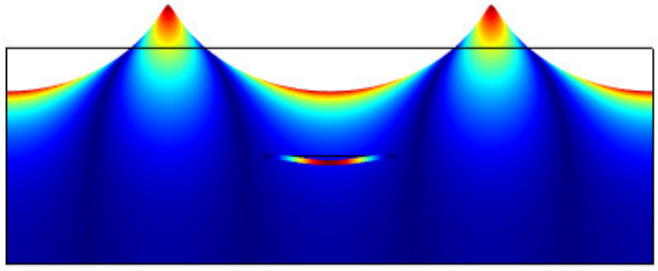
(a) $f = 0.022$ Hz

Figure 15: The second coupled natural frequency and mode shape of beams in water in the case of free surface wave disturbance. The colors pertain to the displacement field of beams and to the absolute acoustic pressure.



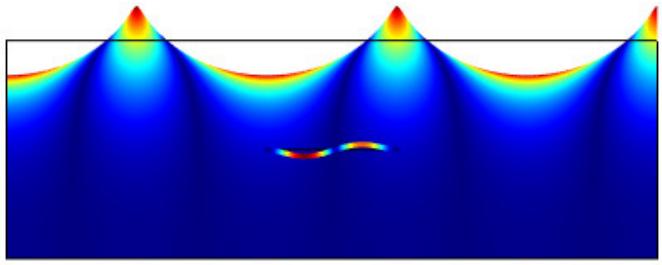
(a) $f = 0.027$ Hz

Figure 16: The third coupled natural frequency and mode shape of beams in water in the case of free surface wave disturbance. The colors pertain to the displacement field of beams and to the absolute acoustic pressure.



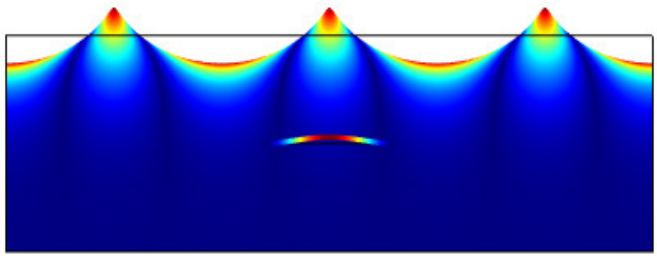
(a) $f = 0.032$ Hz

Figure 17: The fourth fourth coupled natural frequency and mode shape of beams in water in the case of free surface wave disturbance. The colors pertain to the displacement field of beams and to the absolute acoustic pressure.



(a) $f = 0.036$ Hz

Figure 18: The fifth fourth coupled natural frequency and mode shape of beams in water in the case of free surface wave disturbance. The colors pertain to the displacement field of beams and to the absolute acoustic pressure.



(a) $f = 0.039$ Hz

Figure 19: The sixth coupled natural frequency and mode shape of beams in water in the case of free surface wave disturbance. The colors pertain to the displacement field of beams and to the absolute acoustic pressure.

to overcome the difficulty of evaluating the effects of fluid loading. The numerical results calculated using the Comsol Multiphysics FEM Simulation Software are compared with analytical results and show good agreement. It is concluded that the methodology proposed by this paper can be widely applied to most boundary conditions of Euler-Bernoulli beams with infinite number of mass points. This study takes into consideration both fluid loading and concentrated mass loading effects, both of these factors are important in engineering. Numerical results of the natural frequencies and mode shapes of the beams-fluid with or without point masse are presented.

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Appendix B. Matrix elements

The matrix $\mathbf{N}$ in Eq. (12) is defined as follows :

$$\mathbf{N}_{11} = \{a_{11}, a_{12}, a_{13}, a_{14}, 0, 0, 0, 0.\}$$
$$\mathbf{N}_{21} = \{a_{21}, a_{22}, a_{23}, a_{24}, 0, 0, 0, 0.\}$$
$$\mathbf{N}_{31} = \{a_{31}, a_{32}, a_{33}, a_{34}, a_{35}, a_{36}, a_{37}, a_{38}.\}$$
$$\mathbf{N}_{41} = \{a_{41}, a_{42}, a_{43}, a_{44}, a_{45}, a_{46}, a_{47}, a_{48}.\}$$
$$\mathbf{N}_{51} = \{a_{51}, a_{52}, a_{53}, a_{54}, a_{55}, a_{56}, a_{57}, a_{58}.\}$$
$$\mathbf{N}_{61} = \{a_{61}, a_{62}, a_{63}, a_{64}, a_{65}, a_{66}, a_{67}, a_{68}.\}$$
$$\mathbf{N}_{71} = \{0, 0, 0, 0, a_{75}, a_{76}, a_{77}, a_{78}.\}$$
$$\mathbf{N}_{81} = \{0, 0, 0, 0, a_{85}, a_{86}, a_{87}, a_{88}.\}$$

where

$$\begin{aligned}
a_{11} &= u_1(0) \quad , \quad a_{21} = u_1'(0) \\
a_{12} &= u_2(0) \quad , \quad a_{22} = u_2'(0) \\
a_{13} &= u_3(0) \quad , \quad a_{23} = u_3'(0) \\
a_{14} &= u_4(0) \quad , \quad a_{24} = u_4'(0) \\
a_{31} &= u_1(l_1) \quad , \quad a_{41} = u_1'(l_1) \\
a_{32} &= u_2(l_1) \quad , \quad a_{42} = u_2'(l_1) \\
a_{33} &= u_3(l_1) \quad , \quad a_{43} = u_3'(l_1) \\
a_{34} &= u_4(l_1) \quad , \quad a_{44} = u_4'(l_1) \\
a_{35} &= u_1(l_1) \quad , \quad a_{45} = u_1'(l_1) \\
a_{36} &= u_2(l_1) \quad , \quad a_{46} = u_2'(l_1) \\
a_{37} &= u_3(l_1) \quad , \quad a_{47} = u_3'(l_1) \\
a_{38} &= u_4(l_1) \quad , \quad a_{48} = u_4'(l_1) \\
a_{51} &= u_1''(l_1) \quad , \quad a_{61} = EIu_1'''(l_1) + m\omega^2 u_1(l_1) \\
a_{52} &= u_2''(l_1) \quad , \quad a_{62} = EIu_2'''(l_1) + m\omega^2 u_2(l_1) \\
a_{53} &= u_3''(l_1) \quad , \quad a_{63} = EIu_3'''(l_1) + m\omega^2 u_3(l_1) \\
a_{54} &= u_4''(l_1) \quad , \quad a_{64} = EIu_4'''(l_1) + m\omega^2 u_4(l_1) \\
a_{55} &= u_1''(l_1) \quad , \quad a_{65} = EIu_1'''(l_1) \\
a_{56} &= u_2''(l_1) \quad , \quad a_{66} = EIu_2'''(l_1) \\
a_{57} &= u_3''(l_1) \quad , \quad a_{67} = EIu_3'''(l_1) \\
a_{58} &= u_4''(l_1) \quad , \quad a_{68} = EIu_4'''(l_1) \\
a_{75} &= u_1(l) \quad , \quad a_{85} = u_1'(l) \\
a_{76} &= u_2(l) \quad , \quad a_{86} = u_2'(l) \\
a_{77} &= u_3(l) \quad , \quad a_{87} = u_3'(l) \\
a_{78} &= u_4(l) \quad , \quad a_{88} = u_4'(l)
\end{aligned}$$

and using Eq. (9), the quantities $u_i(x)(i = 1, 2, 3, 4)$ are defined as follows

$$u_1(x) = \phi(x) - \beta U \cos(\alpha_n x)$$
$$u_2(x) = \psi(x) - \beta X \cos(\alpha_n x)$$
$$u_3(x) = \chi(x) - \beta Y \cos(\alpha_n x)$$
$$u_4(x) = \varphi(x) - \beta Z \cos(\alpha_n x)$$