A new system formulation for the tolerance analysis of overconstrained mechanisms

A. Dumas[a,b,*], N. Gayton[b], J.-Y. Dantan[a], B. Sudret[c]

[a] *LCFC, Arts et Métiers ParisTech Metz, 4 rue Augustin Fresnel, 57078 Metz Cedex 3, France*
[b] *Clermont Université, IFMA, UMR 6602, Institut Pascal, BP 10448, F-63000 Clermont-Ferrand, France*
[c] *ETH Zürich, Institute of Structural Engineering, Chair of Risk, Safety & Uncertainty Quantification, Wolfgang-Pauli-Strasse 15 CH-8093 Zürich, Switzerland*

Abstract

The goal of tolerance analysis is to verify whether design tolerances enable a mechanism to be functional. The current method consists in computing a probability of failure using Monte Carlo simulation combined with an optimization scheme called at each iteration. This time consuming technique is not appropriate for complex overconstrained systems. This paper proposes a transformation of the current tolerance analysis problem formulation into a parallel system probability assessment problem using the Lagrange dual form of the optimization problem. The number of events being very large, a preliminary selective search algorithm is used to identify the most contributing events to the probability of failure value. The First Order Reliability Method (FORM) for systems is eventually applied to compute the probability of failure at low cost. The proposed method is tested on an overconstrained mechanism modeled in three dimensions. Results are consistent with those obtained with the Monte Carlo simulation and the computing time is significantly reduced.

Keywords: tolerance analysis, Lagrange dual, FORM, system reliability, selective search

*Corresponding author
Email addresses: antoine.dumas@ifma.fr (A. Dumas), nicolas.gayton@ifma.fr (N. Gayton), jean-yves.dantan@ensam.eu (J.-Y. Dantan), sudret@ibk.baug.ethz.ch (B. Sudret)

1. Introduction

Nomenclature	
P_f	Functional failure probability
y_{th}	Threshold value
X	Geometrical deviation = Random variable
g	Gap variable = optimization variable
$C_f \geq 0$	Functional condition
$C_{f,dual} \geq 0$	Functional condition in the dual form
$C \leq 0$	Interface constraints
N_C	Number of interface constraints
N_s	Number of possible situations
$N_{as} \leq N_s$	Number of admissible situations
$N_{ds} \leq N_{as}$	Number of dominant admissible situations

A manufacturing process is not able to provide exactly the same workpieces; indeed, theoretical dimensions of a design product cannot be reached in a repetitive manner (tool wear, operator variability,...). The mechanism behavior is disturbed by geometrical deviations as well as gaps between different parts of the mechanism. Design tolerances are therefore specified on different features of the mechanism to limit the deviations. Tolerance analysis aims at analyzing the impact of these admissible variations on the mechanism behavior. The main stake is to evaluate a quality level of the product during its design stage. The technique used consists of assessing a probability of failure P_f of the mechanism of magnitude around 10^{-6} for large series production. This value represents the probability that a functional condition, $C_f = y_{th} - Y \geq 0$, is not satisfied, where Y is a functional characteristic of the mechanism and y_{th} is a threshold value to not be exceeded.

Tolerance analysis methods must consider the geometrical deviations as random variables whose probabilistic distributions are chosen regarding the manufacturing process [1, 2]. However, gaps between parts or contact points cannot

be modeled by aleatory uncertainty. Gaps belong to the parameter uncertainty category [3] of the epistemic uncertainty which makes difficult the mechanical behavior of this kind of mechanisms to be modeled. Indeed, analyzing isoconstrained or overconstrained mechanisms is different. An assembly which have only its six degrees of freedom fixed in three dimensions (three degrees of freedom in two dimensions) is considered to be an isoconstrained mechanism, usually without gaps. On the contrary, an assembly which have more than six degrees of freedom fixed is considered as an overconstrained mechanism. Gaps allows this kind of mechanism to be assembled although more than six degrees of freedom are fixed. Figure 1 shows a simple isoconstrained mechanism in one dimension where the functional characteristic Y must not exceed a specified threshold. On such a mechanism, the expression of this characteristic Y is a function only of the dimensions x_1 and x_2. This kind of problem is well-defined. In contrast, Figure 2 shows two configurations of an overconstrained mechanism.

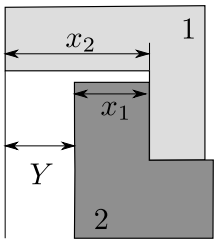

Figure 1: Isostatic mechanism. The functional characteristic Y can be expressed as a function of variables x_1 and x_2.

Now the functional characteristic Y is a function of the random variables and of the gap values. However, following the realization of the random variables, gap values are depending on the location of the contact point between part 1 and part 2. In this case the tolerance analysis problem is overconstrained due to the multiple possible configurations of gaps. Tolerance analysis methods must therefore take into account the worst configurations of gaps to compute the probability of failure. This operation is usually performed using an optimization scheme [1]. This particularity turns out to deal with a system probability

assessment, the transition from one worst configuration to another leading to an abrupt change of the limit-state surface.

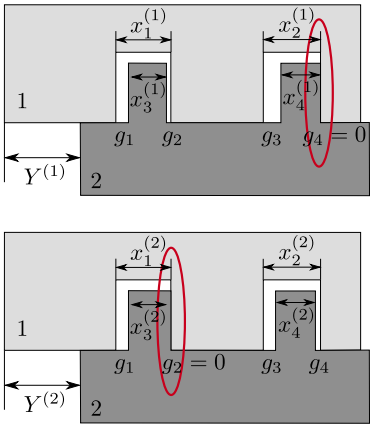

Figure 2: Two worst configurations of gaps in a 1D mechanism. The goal is to find the maximum value of Y. For two given sets of random variable realizations $\left\{\mathbf{x}_i^{(j)}, i = 1, \ldots, 4; j = 1, 2\right\}$, the worst configuration of gaps changes: contact with the right pin for $j = 1$ and contact with the left pin for $j = 2$.

For an isoconstrained mechanism, computing the probability of failure is simple because the functional condition is only expressed as a function of the random variables which describes the geometrical parameters. Classical reliability methods such as Monte Carlo simulation or variance reduction techniques like Importance Sampling [4] can be used to quickly compute the probability of functional failure. For complex non explicit applications, surrogate models replacing the true functional condition may also be used in simulations in order to save time. Numerous techniques used in computer experiments exist in the literature such as quadratic response surfaces [5, 6], Kriging [7, 8], support vector machines [9, 10, 11] and polynomial chaos [12, 13, 14]. Approximation methods like FORM [15, 16] can also be performed. All methods are efficient provided that the problem has a smooth limit-state surface which is the case for an isoconstrained mechanism tolerance analysis problem. For an overconstrained mechanism, these techniques, except the Monte Carlo simulation, can no longer be used because of a piecewise limit-state function coming from the

different configurations of gaps.

This paper intends to propose an efficient method to compute the probability of failure of a tolerance analysis problem in the case of overconstrained mechanisms. The technique is based on a transformation of the tolerance analysis problem formulation using the Lagrange duality property. This operation leads to an auxiliary problem which is free from the optimization step. The solution method includes a selective search algorithm so as to determine the dominant failure situations among the numerous possible ones. The probability of failure is eventually computed using the First Order Reliability Method (FORM) for systems [15, 16].

The paper is organized as follows: section 2 shows the current tolerance analysis problem formulation whose probability is estimated thanks to the Monte Carlo simulation. Section 3 describes the mathematical transformation of the optimization problem into its Lagrange dual. Section 4 is devoted to the comparison of the proposed method with the Monte Carlo simulation on different applications: first, the different transformation steps of the proposed formulation are detailed on a simple academic example. Then the method is applied to an overconstrained industrial application modeled in three dimensions.

2. Tolerance analysis of overconstrained mechanisms

2.1. Problem formulation based on quantifiers

The presence of gaps in overconstrained mechanisms makes the mechanical behavior difficult to model. Gaps are considered as free variables, but they are not free of constraints because interpenetration between two surfaces of two parts of the mechanism cannot be allowed. A set of N_C interface constraints are therefore defined to prevent surfaces from penetrating into each other. Let $\mathbf{X} = \{X_1, \ldots, X_n\}$ be the vector of random variables and $\mathbf{g} = \{g_1, \ldots, g_m\}$ the vector of gaps. Given a realization $\mathbf{x}$ of the random vector $\mathbf{X}$, these constraints are inequations written as follows:

$$\{C_k(\mathbf{x}, \mathbf{g}) \leq 0\}_{k=1,\ldots,N_C} \tag{1}$$

The functional condition equation of the mechanism is expressed as follows:

$$C_f(\mathbf{x}, \mathbf{g}) = y_{th} - Y \geq 0 \tag{2}$$

where $Y = f(\mathbf{x}, \mathbf{g})$ is the response of the system (a parameter such as a gap or a functional characteristic) modeled by a function f characterizing the influences of the deviations and gaps on the mechanism behavior [17].

The universal quantifier "$\forall$" (all) is used to translate the concept that the functional condition must be must respected in all configuration of the mechanism. The definition of the functionality of the mechanism is given by Qureshi *et al.* [1]: "for all admissible gap configurations of the mechanism, the geometrical behavior and the functional requirement are respected". For any realization $\mathbf{x}$ of the random vector describing the geometrical deviations, the mechanism is functional if the following holds:

$$C_f(\mathbf{x}, \mathbf{g}) \geq 0, \forall \mathbf{g} \in \mathbb{R}^m : C_k(\mathbf{x}, \mathbf{g}) \leq 0, \forall \mathbf{k} \in 1, \ldots, N_C \tag{3}$$

The goal of tolerance analysis is to compute a probability of failure. From the previous definition of the functionality, the definition of the non functionality can be formulated as: "there exists at least one admissible configuration for which the functional condition is not respected". That is why the worst configurations of gaps must be considered which is obtained when the minimum value of $C_f(\mathbf{x}, \mathbf{g})$ is found whereas the interface constraints are satisfied. The expression of the probability of failure is given in Eq. (4).

$$P_f = \text{Prob}\left(R(\mathbf{X}) \leq 0\right) \tag{4}$$

where $R(\mathbf{x}) = \min_{\mathbf{g} \in \mathbb{R}^m} C_f(\mathbf{x}, \mathbf{g})$ such that $C_k(\mathbf{x}, \mathbf{g}) \leq 0, \forall \mathbf{k} \in 1, \ldots, N_C$.

2.2. First approach combining Monte Carlo simulation and optimization

A straight forward approach to estimate the probability of failure is based on the Monte Carlo simulation combined with an optimization algorithm. Let us introduce the indicator function $\mathbb{1}_{\min C_f \leq 0}$ being equal to one if $\min_{\mathbf{g}} C_f(\mathbf{x}, \mathbf{g}) \leq$

0 and zero otherwise. The probability of failure estimation is then expressed as follows:

$$\widehat{P}_{fmc} = \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\min C_f \leq 0}\left(\mathbf{x}^{(i)}\right) \tag{5}$$

where $\left\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(N)}\right\}$ is a set of samples from the random vector $\mathbf{X}$. This estimation technique, combined with the optimization scheme to evaluate $\mathbb{1}_{\min C_f \leq 0}\left(\mathbf{x}^{(i)}\right)$ for each (i), is easy to implement and able to provide an accurate estimate of the probability provided that the coefficient of variation of the estimator, given Eq. (6), is small enough (<10%).

$$C.O.V_{\widehat{P}_{fmc}} = \sqrt{\frac{1-\widehat{P}_{fmc}}{N\widehat{P}_{fmc}}} \tag{6}$$

Monte Carlo simulation is the reference method, which can cope with a piecewise continuous limit-state surface. For simple mechanisms, the optimization step is costless although repeated N times. However, for highly overconstrained systems, the optimization step makes the Monte Carlo simulation too much time consuming, especially for small target probabilities. A new method is therefore required in order to analyze complex systems faster.

3. System reliability formulation based on Linear Programming

As previously stated in the introduction, the full failure domain turns out to be a system combinations of several failure domains relative to the different configurations of gaps of the mechanism. Indeed, due to gaps and following the realization of the random variables, several worst configurations of gaps may exist in a mechanism when minimizing $C_f(\mathbf{x}, \mathbf{g})$ [18]. In practice, some configurations often occur but some others may never happen. If several dominant configurations are identified, then several discontinuities appears in the limit-state surface. It represents a change of position of the mechanism parts from a configuration to another. A worst configuration leading to a non functional domain is called a failure situation. In [18], Beaucaire *et al.* propose a system reliability formulation which can be obtained by solving the KKT optimality

conditions of the optimization problem given in Eq. (4). However this formulation leads to define a probability formulation as the union of intersection of events and cannot be used for complex mechanism.

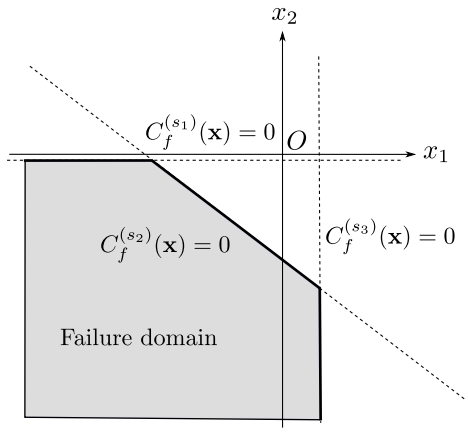

Figure 3: Limit-state surface for a mechanism with two random variables and three failure situations.

Figure 3 shows a limit-state surface for a mechanism with two random variables and three failure situations. It appears clearly that the probability of failure matches the intersection of all failure domains of each situation. The goal of this paper is to demonstrate this assumption from the current tolerance analysis formulation and to propose a procedure to compute the probability of failure P_f based on a system formulation. Subsection 3.1 shows the proposed formulation of the tolerance analysis problem based on the Lagrange dual form of the optimization problem. Subsection 3.2 describes the FORM method for systems. Subsection 3.3 proposes a method in order to reduce the number of intersection events by selecting only dominant failure situations.

3.1. Transformation of the optimization problem into the Lagrange dual form

The tolerance analysis problem formulation is defined as a linear function of gaps [1]. Indeed the functional condition is always defined as a linear combinations of several gap components. Interface constraints are most of the time also written as a linear functions of gaps [1]. However, cylinder type joints lead

to define quadratic interface constraints. In this case, constraints have to be linearized. This linearization of the behavior model has an impact on the probability of failure but this is not the purpose of the present paper. Given these properties, the optimization problem belongs therefore to the linear programming category. In addition, the objective function C_f is assumed differentiable. Given a realization $\mathbf{x} = \{x_1, \ldots, x_n\}$ of the random vector $\mathbf{X}$, let us define the Lagrange function:

$$L(\mathbf{x}, \mathbf{g}, \boldsymbol{\lambda}) = C_f(\mathbf{x}, \mathbf{g}) + \sum_{k=1}^{N_C} \lambda_k C_k(\mathbf{x}, \mathbf{g}) \tag{7}$$

where $\boldsymbol{\lambda}$ are the N_C Lagrange multipliers [19].

Proposition 3.1. *In the present case, $\mathbf{g}^*$ is an optimal solution of the optimization problem if and only if there exists $\boldsymbol{\lambda}^* \geq 0$ such that $(\mathbf{g}^*, \boldsymbol{\lambda}^*)$ is a saddle point of the Lagrange function in Eq. (7). In particular, the optimization problem has a solution if and only if L has saddle points, and if it does, then its Lagrange dual problem has a solution whose optimal values are equal.*

Let us consider a Linear Programming problem, characterizing a tolerance analysis problem, which is written in a matrix form as follows:

$$\begin{aligned} \min_{\mathbf{g}} \quad & C_f(\mathbf{x}, \mathbf{g}) = \mathbf{a}^T\mathbf{x} + \mathbf{b}^T\mathbf{g} + c_0 \\ \text{subject to} \quad & C_k(\mathbf{x}, \mathbf{g}) = \mathbf{d}_k^T\mathbf{x} - \mathbf{e}_k^T\mathbf{g} + c_k \leq 0 \end{aligned} \tag{8}$$

for $k = 1, \ldots, N_C$ and for $\mathbf{g} \in \mathbb{R}^m$. The Lagrange function is given by:

$$\begin{aligned} L(\mathbf{x}, \mathbf{g}, \boldsymbol{\lambda}) &= C_f(\mathbf{x}, \mathbf{g}) + \sum_{k=1}^{N_C} \lambda_k C_k(\mathbf{x}, \mathbf{g}) \\ &= \left[\mathbf{b} - \sum_{k=1}^{N_C} \lambda_k \mathbf{e}_k\right]^T \mathbf{g} + \sum_{k=1}^{N_C} \lambda_k \left(\mathbf{d}_k^T\mathbf{x} + c_k\right) + \mathbf{a}^T\mathbf{x} + c_0 \end{aligned} \tag{9}$$

Finally, the Lagrange dual problem reads:

$$\begin{aligned} \max_{\boldsymbol{\lambda}} \quad & C_{f,dual}(\mathbf{x}, \boldsymbol{\lambda}) = \sum_{k=1}^{N_C} \lambda_k \left(\mathbf{d}_k^T\mathbf{x} + c_k\right) + \mathbf{a}^T\mathbf{x} + c_0 \\ \text{subject to} \quad & \sum_{k=1}^{N_C} \lambda_k \mathbf{e}_k = \mathbf{b}; \boldsymbol{\lambda} \geq 0 \end{aligned} \tag{10}$$

The expression of the dual form allows the objective function to be no more a function of gaps. In addition, according to the proposition 3.1, if an optimal solution exists, then both optimization problems in Eqs. (8) and (10) have the same solution. The optimization constraints are now m equality equations to be satisfied and the Lagrange multipliers must be positive. The number of interface constraints N_C being greater than the dimension m, the dual optimization problem is under constraints: there are more Lagrange multipliers $\boldsymbol{\lambda}$ than equality constraints. In order to find the admissible Lagrange multipliers satisfying the constraints, all possible combinations must be tested. However, some combinations are impossible. Table 1 lists several impossible situations.

Table 1: List of impossible combinations of Lagrange multiplier values.

Impossible situation	Physical interpretation
All $\boldsymbol{\lambda}$ equal to 0 because $\mathbf{b} \neq 0$.	This is equivalent to consider that there is no contact between the mechanism parts because no interface constraints can be equal to zero.
All $\boldsymbol{\lambda}$ are different from 0 because $N_C >> m$.	This is equivalent to consider that there are contacts in every possible points of the mechanism simultaneously because all interface constraints must be equal to zero.

In fact, the number of non zero Lagrange multipliers $\boldsymbol{\lambda}_{s_k} = \{\lambda_{s_k^{(1)}}, \ldots, \lambda_{s_k^{(m)}}\}$ must be equal to the number of gaps to be found, where s_k contains the indices of the Lagrange multipliers different from zero of the k^{th} situation. So, the number of possible situations is equal to $N_s = \binom{N_C}{m}$, but all situations may not be admissible regarding the optimization constraints, given in Eq. (10). Let N_{as} be the number of the admissible situations of Lagrange multipliers satisfying the given constraints and $C_{f,dual}^{(s_k)}(\mathbf{x}) = C_{f,dual}(\mathbf{x}, \boldsymbol{\lambda}_{s_k})$ be the expression of the dual functional condition relative to the combination s_k. There are then N_{as}

admissible solutions to the maximization problem, in Eq. (10), which can now be written as follows:

$$C^*_{f,dual}(\mathbf{x}) = \max_{i=\{1,\ldots,N_{as}\}} \left(\left\{ C^{(s_i)}_{f,dual}(\mathbf{x}) \right\} \right) \tag{11}$$

In addition, according to the proposition 3.1, the optimal values of the primal and dual optimization problem are equal, *i.e.* both expressions of the functional condition are equal:

$$C^*_f(\mathbf{x}) = C^*_{f,dual}(\mathbf{x}) = \max_{i=\{1,\ldots,N_{as}\}} \left(\left\{ C^{(s_i)}_f(\mathbf{x}) \right\} \right) \tag{12}$$

Finally, the probability of failure is simply expressed as follows:

$$\begin{aligned} P_f &= \text{Prob}\left(\max_i \left(\left\{ C^{(s_i)}_f(\mathbf{X}) \right\} \right) \leq 0 \right) && (13) \\ &= \text{Prob}\left(\bigcap_{i=1}^{N_{as}} C^{(s_i)}_f(\mathbf{X}) \leq 0 \right) && (14) \\ &= \text{Prob}\left(\left\{ C^{(s_1)}_f(\mathbf{X}) \leq 0 \right\} \cap \ldots \cap \left\{ C^{(s_{N_{as}})}_f(\mathbf{X}) \leq 0 \right\} \right) && (15) \end{aligned}$$

The above transformation procedure is described in detail in a simple example in Subsection 4.1. This new formulation of the tolerance analysis problem as a simple intersection of events allows the efficient FORM method for systems to be applied to compute the probability of failure. This method is explained in the following Subsection.

3.2. The FORM method for systems

The goal is to compute the probability from the multivariate Gaussian probability density function ϕ_n, with the hypothesis of a first order approximation on each situation [15, 16]. Given N_{as} admissible failure situations, the parallel system failure probability, see Figure 4, is expressed as follows:

$$P_f = \text{Prob}\left(\bigcap_{k=1}^{N_{as}} C^{(s_k)}_f(\mathbf{X}) \leq 0 \right) \tag{16}$$

The reliability indices β_{s_k} and direction cosines $\boldsymbol{\alpha}_{s_k}$, for $k = 1, \ldots, N_{as}$, of each situation can be computed using the Hasofer-Lind-Rackwitz-Fiessler algorithm

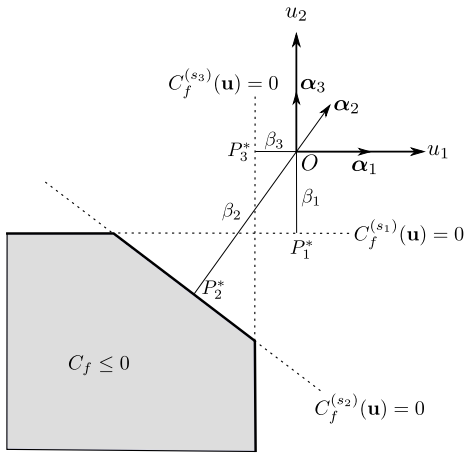

Figure 4: The FORM method for systems in the standard space with three failure domains.

[20] or the improved version [21]. In addition the matrix $[\boldsymbol{\rho}]$ of the limit-state correlation coefficients is gathered from the direction cosines:

$$\rho_{s_k s_j} = \boldsymbol{\alpha}_{s_k} \cdot \boldsymbol{\alpha}_{s_j} \tag{17}$$

The expression of the failure probability is given in Eq. (18).

$$\begin{aligned} P_f &= \text{Prob}\left(\bigcap_{k=1}^{N_{as}} C_f^{(s_k)}(\mathbf{X}) \leq 0\right) \\ &= \Phi_{N_{as}}\left(-\boldsymbol{\beta}; [\boldsymbol{\rho}]\right) \end{aligned} \tag{18}$$

where $\Phi_{N_{as}}$ is the multivariate Gaussian cumulative distribution function. The numerical value of this function can be evaluated using the method proposed by Genz [22]. In addition, a confidence interval on the result is provided with no extra numerical cost.

3.3. Reduction of the number of intersections

The admissible failure situations may be found by computing the Lagrange multipliers in all possible cases. This operation can be achieved if a small number of possibilities exist, *e.g.* $N_s \leq 10$. However for complex systems, the number of cases may exceed several thousands making this operation impossible. In addition, some admissible situations may have no influence on the failure

probability value. In both cases, finding and considering the N_{as} admissible situations is not conceivable. A search algorithm is therefore required in order to select only a number N_{ds} of dominant failure situations. The probability of failure is now expressed as follows:

$$P_f = \Phi_{N_{ds}}\left(-\boldsymbol{\beta}; [\boldsymbol{\rho}]\right) \tag{19}$$

Kim *et al.* [23] propose a simulation-based selective search algorithm which uses a genetic algorithm. It is specially developed to find dominant failure modes of a structure among a large number of possible modes. In the present paper, the failure modes correspond to the failure situations. Considering the standard space of the random variables, the dominant failure situations are located close to the origin. Kim *et al.* propose an outward search in the standard space, from points on a hypersphere near the origin to the points with a larger distance. For a given hypersphere radius, several search directions in the standard space are pseudo randomly defined. Points leading to a failure situation are saved as elite chromosomes and an evolutionary operation is performed in order to find additional failure situations in the same area (crossover) or in the opposite direction (mutation). The evolutionary operation is repeated several times until no new failure situations are found. The hypersphere radius is then increased in order to search further away in the standard space. Once the hypersphere radius is great enough or sufficient failure situations have been found, the algorithm stops. The full algorithm is well described in [23].

4. Applications

A simple academic example is proposed as a first illustration in Subsection 4.1. In particular, the proposed formulation with the transformation steps into the Lagrange dual form is detailed. Subsection 4.2 shows the application of the method to an industrial application. Results are compared with the reference results obtained with the Monte Carlo simulation combined with the optimization scheme.

4.1. Illustration on a simple academic example

The optimization problem characterizing a pseudo behavior model in its primal form is defined in Eq. (20). The functionality is ensured when $C_f \geq 0$. The goal is to compute the probability $P_f = \text{Prob}(\min_{\mathbf{g}} C_f(X_1, X_2, \mathbf{g}) \leq 0)$ where the random variables X_1 and X_2 follow a normal distribution $\mathcal{N}(\boldsymbol{\mu}_X, \boldsymbol{\sigma}_X)$ whose parameters are given in Table 2. Three different set of parameters are used in order to change the order of magnitude of the probability failure.

$$\begin{array}{rll} \min & C_f(\mathbf{x}, \mathbf{g}) = x_1 + x_2 + 1 + g_1 + 2g_2 & \\ \text{subject to} & C_1(\mathbf{x}, \mathbf{g}) = x_1 - 1 - g_1 & \leq 0 \\ & C_2(\mathbf{x}, \mathbf{g}) = x_2 + 2 - g_2 & \leq 0 \\ & C_3(\mathbf{x}, \mathbf{g}) = x_1 - x_2 - 2g_1 & \leq 0 \\ & C_4(\mathbf{x}, \mathbf{g}) = 2x_1 + x_2 + g_1 - 2g_2 & \leq 0 \end{array} \tag{20}$$

Table 2: The three set of parameters of the normal random variables for the academic example.

	μ_X	σ_X		
Set of parameters	All	1	2	3
X_1	0	1	0.5	0.4
X_2	0	1	0.5	0.4

From the definition of the Lagrange dual problem given in Eq. (10), the dual form of this optimization problem is written as follows:

$$\begin{array}{rll} \max\limits_{\boldsymbol{\lambda}} & C_{f,dual}(\mathbf{x}, \boldsymbol{\lambda}) = & \lambda_1(x_1 - 1) + \lambda_2(x_2 + 2) + \lambda_3(x_1 - x_2) \\ & & +\lambda_4(2x_1 + x_2) + x_1 + x_2 + 1 \\ \text{subject to} & \lambda_1 + 2\lambda_3 - \lambda_4 = 1 & \\ & \lambda_2 + 2\lambda_4 = 2 & \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 & \end{array} \tag{21}$$

The number of Lagrange multipliers $\boldsymbol{\lambda}$ is greater than the number of equality constraints so the different possibilities where two λ are different from zero (and the others are set to zero) must be tested, *i.e.* a number of possible situations

$N_s = \binom{4}{2} = 6$. Lagrange multipliers values $\boldsymbol{\lambda}$ are computed solving the linear system of equality constraints. Table 3 lists these possible situations and shows if the result is admissible or not.

Table 3: List of possible combinations to compute Lagrange multiplier values.

Situation hypothesis		Result	Expression of C_f
s_1 :	$\lambda_{3,4} \neq 0$ $\lambda_{1,2} = 0$	$\lambda_3 = 1$ $\lambda_4 = 1$	$C_f^{s_1} = 4x_1 + x_2 + 1$
s_2 :	$\lambda_{2,4} \neq 0$ $\lambda_{1,3} = 0$	$\lambda_4 = -1 < 0$ $\lambda_2 = 4$	It is not an optimum
s_3 :	$\lambda_{2,3} \neq 0$ $\lambda_{1,4} = 0$	$\lambda_2 = 2$ $\lambda_3 = 1/2$	$C_f^{s_3} = \dfrac{3x_1 + 5x_2}{2} + 5$
s_4 :	$\lambda_{1,4} \neq 0$ $\lambda_{2,3} = 0$	$\lambda_1 = 2$ $\lambda_4 = 1$	$C_f^{s_4} = 5x_1 + 2x_2 - 1$
s_5 :	$\lambda_{1,3} \neq 0$ $\lambda_{2,4} = 0$	Constraint of Eq. (21) $\Rightarrow 2 = 0$	Impossible
s_6 :	$\lambda_{1,2} \neq 0$ $\lambda_{3,4} = 0$	$\lambda_1 = 1$ $\lambda_2 = 2$	$C_f^{s_6} = 2x_1 + 3x_2 + 4$

The admissible failure situations are found in this case by computing the Lagrange multipliers in order to describe in detail why some configurations are not admissible. The selective search algorithm, see Subsection 3.3, can also be used. From Table 3, the number of admissible situations is $N_{as} = 4$. The optimization problem is now written as follows:

$$\begin{aligned} C_f^*(\mathbf{x}) &= \max\left(C_f^{s_1}(\mathbf{x}), C_f^{s_3}(\mathbf{x}), C_f^{s_4}(\mathbf{x}), C_f^{s_6}(\mathbf{x})\right) \\ &= \max\left(4x_1 + x_2 + 1, \frac{3x_1 + 5x_2}{2} + 5, 5x_1 + 2x_2 - 1, 2x_1 + 3x_2 + 4\right) \end{aligned} \tag{22}$$

The associated probability of failure is only a function of random variables X_1

and X_2, as seen from the following expression:

$$P_f = \text{Prob}\left(\begin{array}{l} \{4X_1 + X_2 + 1 \leq 0\} \cap \left\{ \dfrac{3X_1 + 5X_2}{2} + 5 \leq 0 \right\} \\ \cap \{5X_1 + 2X_2 - 1 \leq 0\} \cap \{2X_1 + 3X_2 + 4 \leq 0\} \end{array} \right) \tag{23}$$

The FORM method is applied on each failure situation of the previous formulation. In this case of linear functions with normal random variables, the FORM method provides exact results of each reliability index. Results obtained for the first set of parameters are shown in Table 4. The correlation matrix associated with this problem is given in Eq. (24).

Table 4: Reliability indices obtained for the first set of parameters for the academic example.

Failure situation	s_1	s_3	s_4	s_6
β	0.24	1.71	-0.19	1.12

$$[\boldsymbol{\rho}] = \begin{bmatrix} 1 & 0.707 & 0.998 & 0.796 \\ 0.707 & 1 & 0.739 & 0.991 \\ 0.998 & 0.739 & 1 & 0.824 \\ 0.796 & 0.991 & 0.824 & 1 \end{bmatrix} \tag{24}$$

Given these values, the probability of failure can be computed using the multivariate Gaussian cumulative distribution function in four dimensions:

$$P_f = \Phi_4 \left(- \{\beta_{s_1}, \beta_{s_3}, \beta_{s_4}, \beta_{s_6}\} ; [\boldsymbol{\rho}] \right) \tag{25}$$

The comparison of the results obtained with the Monte Carlo simulation (for a C.O.V $\approx 5\%$) and the FORM method for systems are shown in Table 5 for the three set of parameters. The probability of failure is expressed in parts per million (ppm). The smallest the probability, the longer the Monte Carlo simulation to be accurate enough. In contrast, using the proposed formulation and solution method, the computing time is small and stable. Results provided by the FORM method for systems are consistent with those obtained by the Monte Carlo simulation. In addition, the selective search algorithm appears useful because only two failure situations are selected as dominant situations,

which allows again to save time. Indeed, only the failure situations s_1 and s_3 are required to compute the probability of failure. As shown in Figure 5, the failure domain is the intersection of two failure domains corresponding with the situations s_1 and s_3.

Table 5: Comparison between Monte Carlo simulation and FORM method for systems results for the academic example.

	Monte Carlo	FORM system
	42300	41214
P_d (x10^{-6})	307	301
	9.06	9.03
	8050	
95% C.I. (x10^{-6})	57	
	1.7	
	4 s	0.7 s
Computing time	11 min	0.7 s
	6.2 h	0.6 s

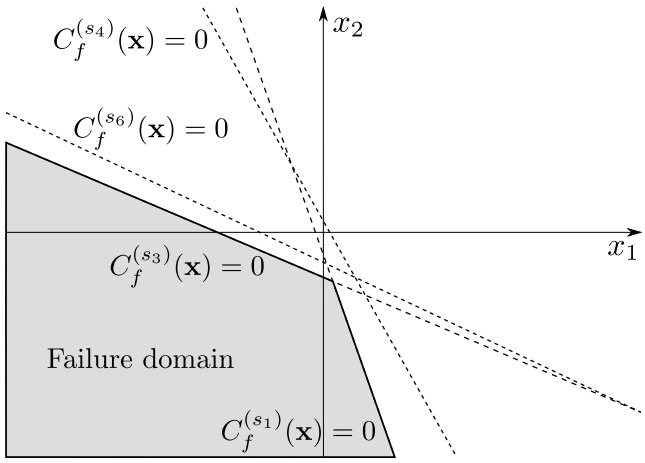

Figure 5: Failure domain of the academic example. The failure domain is the intersection of two failure domains of situations s_1 and s_3.

4.2. Industrial application

The application is based on a gear pump, see Figure 6, which has two parts positioned with two pins. The positioning of these two parts has an influence on the angle of both gear axes. The functionality of the pump can be reduced if the assembly precision of the parts is insufficient. Based on this pump, a simplified overconstrained mechanism is studied. Figure 7 shows the mechanism with amplified gaps between parts. The functional condition concerns the deviation of the point G of part (1) with respect to part (2). This point G can be seen as a functional point that is representative of one axis of the gear pump.

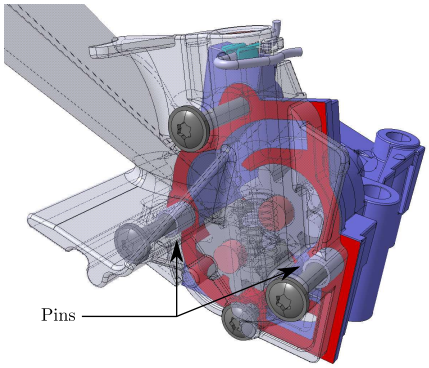

Figure 6: Full pump view.

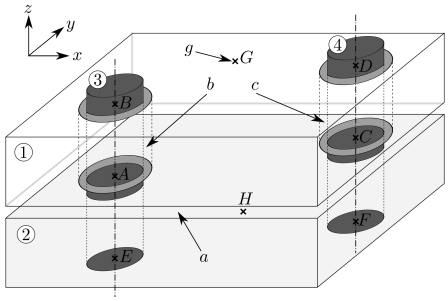

Figure 7: Simplified mechanism in 3D.

Characteristics of the mathematical behavior model are listed below:

- 38 random variables following a Gaussian distribution $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}_{\boldsymbol{X}}, \boldsymbol{\sigma}_{\boldsymbol{X}})$. Values of all parameters are given in Appendix A.
- 3 gap variables $\mathbf{g}$ which are the optimizations parameters.
- 4 quadratic interface constraints which give $N_C = 160$ interface constraints after applying a linearization procedure.
- $N_s = \binom{160}{3} = 669920$ possible situations.

In order to show the efficiency of the proposed formulation, different orders of magnitude of the probability of failure are intended to be reached. Probabilities of failure are expressed in parts per million (ppm). Table 6 shows the

numerical results obtained for three levels of probability. The maximum number of dominant failure situations, N_{ds}, found by the selective search algorithm is set to 5, 100 and 200. A number of failure situations too small leads to a loss of accuracy of the probability of failure whereas a too large number leads to a waste of time in calculation because there is no need to increase the precision on the result. In cases with a large N_{ds}, it can be seen that the FORM method for systems allows to get accurate results that are identical to the reference results with the Monte Carlo simulation. The obtained values of the confidence interval are small enough to ensure an accurate result. In addition, reaching low probabilities with precision is also possible within an acceptable computing time. Results are obtained in less than two minutes whereas the Monte Carlo simulation requires several hours. Indeed, low probabilities lead to define a large Monte Carlo population so as to yield an accurate estimation of the probability of failure, thus greatly increasing the computing time because of the optimization step.

Table 6: Comparison between Monte Carlo simulation and FORM method for systems results for the industrial application.

	Monte Carlo	FORM system		
N_{ds}		5	100	200
P_d (x10^{-6})	17933	23889	18239	18289
	155	282	156	154
	9.4	16	9.3	9.1
95% C.I.	3065			
	28			
	1.7			
Computing time	3 min	22 s	66 s	2.15 min
	5.1 h	23 s	67 s	2.26 min
	3.5 d	26 s	73 s	2.35 min

5. Conclusion

Considering that a mechanism behavior of a product is disturbed by admissible geometrical deviations and gaps between surfaces, it is natural to dispose of a tool able to evaluate the quality level of a designed mechanism. The tolerance analysis method takes into account the specified design tolerances in order to determine whether they ensure a mechanism to be assembled and functional. Evaluating the quality level turns out to be an estimation of a probability of failure which must be done as fast and accurate as possible. The standard method, which combines Monte Carlo simulation and an optimization scheme, is not able to provide accurate result and small values of ppm in a reasonable computing time for complex overconstrained mechanism. This paper proposes a new procedure to deal with a functional tolerance analysis.

The proposed procedure is based on a system formulation of the problem so that a system reliability method can be used. This formulation is obtained using the Lagrange dual form of the original optimization problem. The system formulation of the problem is defined as a simple intersection of events. A simulation-based selective search algorithm is used in order to find only the dominant events. The probability of failure is then quickly estimated using the FORM method for systems.

An academic example is used to detail all the transformation procedure. Then the proposed method is applied on an industrial application modeled in three dimensions. Results show that the FORM method for systems provides accurate result much faster than with the Monte Carlo simulation. In addition, the procedure is able to handle problems where low probabilities are intended to be reached. The application remains simple enough to be able to perform a Monte Carlo simulation so as to compare results. The major advantage of the proposed method is for highly complex systems, a functional condition computed using a finite element modeling for instance, in this case Monte Carlo simulation cannot be applied anymore. The proposed procedure is developed to solve tolerance analysis problems, though it is conceivable that other research

fields may have problems written as the original formulation. In this case, the transformation of the formulation into a system formulation may also be applied.

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Appendix A. Parameters of the normal random variables of the industrial application

	μ_X	σ_X		
Set of parameters	All	1	2	3
y_{th}		0.25	0.28	0.29
X_1	20	0.06	0.03	0.02
X_2	19.8	0.06	0.03	0.02
X_3	20	0.06	0.03	0.02
X_4	19.8	0.06	0.03	0.02
X_5	0	0.01	0.01	0.01
X_6	0	0.01	0.01	0.01
X_7	0	0.01	0.01	0.01
X_8	0	0.01	0.01	0.01
X_9	0	0.01	0.01	0.01
X_{10}	0	0.01	0.01	0.01
X_{11}	0	0.01	0.01	0.01
X_{12}	0	0.01	0.01	0.01
X_{13}	0	0.01	0.01	0.01
X_{14}	0	0.01	0.01	0.01
X_{15}	0	0.01	0.01	0.01
X_{16}	0	0.01	0.01	0.01
X_{17}	0	0.01	0.01	0.01

	μ_X	σ_X		
Set of parameters	All	1	2	3
X_{18}	0	0.01	0.01	0.01
X_{19}	0	0.01	0.01	0.01
X_{20}	0	0.01	0.01	0.01
X_{21}	0	0.01	0.01	0.01
X_{22}	0	0.01	0.01	0.01
X_{23}	0	0.01	0.01	0.01
X_{24}	0	0.01	0.01	0.01
X_{25}	0	0.01	0.01	0.01
X_{26}	0	0.01	0.01	0.01
X_{27}	0	0.001	0.001	0.001
X_{28}	0	0.001	0.001	0.001
X_{29}	0	0.001	0.001	0.001
X_{30}	0	0.01	0.01	0.01
X_{31}	0	0.01	0.01	0.01
X_{32}	0	0.01	0.01	0.01
X_{33}	0	0.001	0.001	0.001
X_{34}	0	0.001	0.001	0.001
X_{35}	0	0.001	0.001	0.001
X_{36}	0	0.01	0.01	0.01
X_{37}	0	0.01	0.01	0.01
X_{38}	0	0.01	0.01	0.01

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