Solving Magnetodynamic Problems via Normal Form Method

Nnaemeka S. Ugwuanyi[1, 3], Stéphane Clénet[1], Xavier Kestelyn[1], *Member, IEEE*, and Olivier Thomas[2]
[1]Univ. Lille, Arts et Metiers Institute of Technology, Centrale Lille, Junia, ULR 2697 – L2EP, F-59000 Lille, France, nnaemeka.ugwuanyi@univ-lille.fr.
[2]Arts et Métiers Institute of Technology – LISPEN, F-59000 Lille, France.
[3]Department of Electrical/Electronic Engineering, AE-FUNAI, P.M.B. 1010, Abakaliki 480214, Nigeria.

Closed-form formulations are difficult to find when the material behavior law is nonlinear. A linear approximation, on the other hand, has a very narrow range of validity. In this communication, the Normal Form (NF) method is employed to solve a 1-D nonlinear magnetodynamic problem. The discrete model is formulated in a state-space form suitable for NF applications. The resulting system is then expanded on a linear mode basis to cubic order. Analytical solutions are obtained using the NF technique and compared to traditional solutions. The results show the cubic polynomial adequately approximates the problem, and the NF solution is valid for some range of magnetic field intensity.

***Index Terms*—Analytical solutions, computational electromagnetism, Eddy currents, diffusion problems, normal form method.**

I. Introduction

THE need to control the efficiency of electrical devices at the design stage is now a significant consideration. Consequently, all kinds of losses, including eddy current losses, must be correctly estimated. Since eddy current losses in lamination steel are affected by multiple factors such as the $B(H)$ curve, conductivity, frequency, etc., multi-parameter models for evaluating the losses are valuable during design. As a result, closed-form formulations of these losses become increasingly important.

However, there is no analytical expression to determine the eddy current losses considering a nonlinear behavior of the $B(H)$ curve, so a 1-D magnetodynamic problem is handled numerically, which is time-consuming. Furthermore, with closed-form formulations in view, the numerical technique becomes insufficient. The Normal Form (NF) method is a promising tool not yet explored in computational electromagnetism to solve nonlinear problems. The NF method is applied in mechanical engineering to derive reduced-order models, especially by defining nonlinear modes, an extension to the nonlinear regime of the natural modes of vibration, thanks to invariant manifolds of the phase space [1], [2], [3]. More recently, it is being applied to reduce the size of finite element models [4]. It is also applied to study the nonlinear behavior of stressed power systems [5], [6]. It is attractive because it leads to an analytical (closed-form) expression for the solution. Applying this method to solve a magnetodynamic problem, we can expect to derive a closed-form expressions, for example, instantaneous losses.

In this contribution, the application of the NF technique to solve a nonlinear magnetodynamic problem is studied. The NF application requires the equation to solve expressed in a state-space form, with the nonlinear terms polynomials. The original problem is introduced first. Next, the state-space representation with nonlinear polynomial terms is derived following by the NF representation. Finally, the results obtained by the nonlinear state-space and the NF models are compared to the ones obtained by the original problem.

II. A 1-D Magnetodynamic Problem

A. Problem to solve

We consider a 1-D lamination model of thickness e along the Ox axis, and infinite along the Oy and Oz axes. The field distribution is symmetric and the conductivity σ is assumed constant. A magnetic field $H_s(t) = H_{max}\sin(\omega t)$ is imposed on the surface of the lamination, where f is the frequency and $\omega = 2\pi f$. The magnetic field $H(x,t)$ verifies the following equation:

$$\frac{d^2H(x,t)}{dx^2} = -\sigma\frac{dB}{dt}[H(x,t)] \quad \text{with} \quad H(\pm\frac{e}{2},t) = H_s(t). \tag{1}$$

The preceding problem has a closed-form solution when the $B(H)$ curve is linear. The losses can then be determined, precisely and very quickly. However, when the $B(H)$ curve is nonlinear, (1) has no analytical solution. An approximation of the exact solution is then obtained by applying a numerical method like the Finite Element, the Finite Difference, etc.

B. Discrete and state-space formulations

Let N_x be the number of nodes for the space discretization and $\mathbf{H}(t)$ be the $N_x \times 1$ vector of magnetic field $H(x,t)$ values at the nodes. Applying the Finite Difference method we get:

$$\mathbf{MH}(t) = -\mathbf{K}[\mathbf{H}(t)]\frac{d\mathbf{H}(t)}{dt} + \mathbf{H_s}(t), \tag{2}$$

where $\mathbf{M}$ and $\mathbf{K}$ are $N_x \times N_x$ matrices, $\mathbf{H_s}$ is a vector accounting for the boundary conditions on the surface of the lamination (see (1)). The expression $\mathbf{K}[\mathbf{H}(t)]$ implies that the matrix $\mathbf{K}$ is a function of $\mathbf{H}(t)$. The state-space representation can be written more concisely as:

$$\frac{d\mathbf{H}(t)}{dt} = \mathbf{f}(\mathbf{H}(t),t) + \mathbf{F}_{ext}(t), \tag{3}$$

where the expression of $\mathbf{f}(\mathbf{H},t) = -\mathbf{K}[\mathbf{H}(t)]^{-1}\mathbf{MH}(t)$, and $\mathbf{F}_{ext}(t) = \mathbf{K}[\mathbf{H}(t)]^{-1}\mathbf{H_s}(t)$. Applying a numerical scheme like the Newton-Raphson method, a nonlinear problem is solved at each time step t_i, $i \in [1, N_t]$ to obtain $\mathbf{H}(t_i)$, an approximation of the vector $\mathbf{H}(t)$.

One way to get a closed-form solution of (3) is to linearize it around the origin. However, as we will see later, the validity range of a linear approximation is rather small compared to the range of variation of the magnitude of the magnetic field, H_{max}. A higher-order polynomial expansion can adequately approximate (3). Then, by applying the Normal Form method, the new nonlinear problem based on a high-order polynomial expansion can be transformed into a linear problem. A closed-form solution of this linear problem can be obtained, leading to a closed-form of an approximation of our initial nonlinear problem (1). To calculate the eddy current losses in an electrical machine defined by a Finite Element model for example, we have to compute the losses on each element of the mesh of parts made of lamination. The usual method consist in solving the 1-D problem in (2) on each element with the evolution of the magnetic field $\mathbf{H_s}$ on the element. This operation could take time if the number of elements is large. The NF technique may provide a closed-form solution to calculate the losses, thereby reducing the calculation on each element and hence the total losses. The NF technique is discussed in the following section.

III. Normal Form Theory

In (3), $\mathbf{f}$ is a smooth vector function of the entries, $\mathbf{H}(t)$. As noted earlier, the system in (3) can be approximated by a Taylor expansion around the origin. However, the presence of a harmonic, time-dependent forcing term, $\mathbf{F}_{ext}(\mathbf{t})$, renders the standard NF application impracticable because the definitions of time-dependent invariant manifolds are required [2]. To ease this challenge, we invoke the intuitive state-augmentation method proposed in [7] to convert (3) to an autonomous system $\frac{d\tilde{\mathbf{H}}}{dt} = \tilde{\mathbf{f}}(\tilde{\mathbf{H}})$, free of a time-dependent forcing term. $\tilde{\mathbf{f}}$ is the vectorial function of the augmented state vector, $\tilde{\mathbf{H}}$. The state-augmentation procedure is described next.

A. State-augmentation for realizing an autonomous system

Beginning with (2), let us define a new state variable X_1 verifying the following equations.

$$\frac{d^2X_1}{dt^2} = -\omega^2 X_1\ , \quad X_1(0) = 0, \quad \left.\frac{dX_1}{dt}\right|_{t=0} = \omega H_{max}. \tag{4}$$

Define another state $X_2 = \frac{dX_1}{dt}$. Therefore,

$$\frac{dX_1}{dt} = X_2, \quad X_2(0) = \omega H_{max}, \tag{5a}$$

$$\frac{dX_2}{dt} = -\omega^2 X_1, \quad X_1(0) = 0. \tag{5b}$$

The system of (5) is written in a state variable form, and the solution $X_1 = H_{max}\sin\omega t$ corresponds to the source term we want to impose on the boundary. Equations (5) are used to augment (2) to make the system autonomous and fit for standard NF applications. The augmented system now has $N_{aug} = N_x + 2$ differential equations as follows:

$$\begin{bmatrix} \frac{d\mathbf{H}}{dt} \\ \\ \frac{d\mathbf{X}}{dt} \end{bmatrix} = \begin{bmatrix} -\mathbf{K}[\mathbf{H}(t)]^{-1}\mathbf{M} & \mathbf{K}[\mathbf{H}(t)]^{-1}\mathbf{M_c} \\ \\ & \mathbf{M_r} \end{bmatrix} \begin{bmatrix} \mathbf{H} \\ \\ \mathbf{X} \end{bmatrix}, \tag{6}$$

where $\mathbf{X} = [X_1 \ \ X_2]^T$, $\mathbf{M_c}$ is an N_x-by-2 matrix with $M_{c_{(1,1)}} = 1$ and other entries zero, $\mathbf{M_r}$ is a 2-by-$N_x + 2$ matrix with $M_{r_{(1,N_x+2)}} = 1$, $M_{r_{(2,N_x+1)}} = -\omega^2$ and other entries zero. Note also that the forcing term is applicable only to the first row. System (6) can now be written more compactly as

$$\frac{d\tilde{\mathbf{H}}}{dt} = \tilde{\mathbf{f}}(\tilde{\mathbf{H}}), \tag{7}$$

which can be Taylor-expanded around $\tilde{\mathbf{H}}_0 = \begin{bmatrix} \mathbf{H_0} & \mathbf{X_0} \end{bmatrix}^T$. $\mathbf{H_0}$ is an $1 \times N_x$ zero vector and $\mathbf{X_0} = \begin{bmatrix} 0 & \omega H_{max} \end{bmatrix}$.

B. Taylor expansion

Applying a Taylor expansion to each equation of the system (6) and truncating at the third order, leads to a polynomial expression of the time derivative $\frac{d\tilde{H}}{dt}$. The i-th entry of the $\frac{d\tilde{H}}{dt}$ is given by:

$$\begin{aligned} \frac{d\tilde{H}_i}{dt} = \sum_{j=1}^{N_{aug}} A_{ij}\tilde{H}_j + \sum_{j=1}^{N_{aug}}\sum_{k=1}^{N_{aug}} F2_{ijk}\tilde{H}_j\tilde{H}_k + \dots \\ \sum_{j=1}^{N_{aug}}\sum_{k=1}^{N_{aug}}\sum_{l=1}^{N_{aug}} F3_{ijkl}\tilde{H}_j\tilde{H}_k\tilde{H}_l, \end{aligned} \tag{8}$$

where $\tilde{H}_i$ is the i-th component of vector $\tilde{\mathbf{H}}$ and, for all $i, j, k, l = 1, 2, \dots N_{aug}$:

$$A_{ij} = \left.\frac{\partial \tilde{f}_i}{\partial \tilde{H}_j}\right|_{\tilde{\mathbf{H}}=\tilde{\mathbf{H}}_\mathbf{0}}, \quad F2_{ijk} = \left.\frac{1}{2}\frac{\partial^2 \tilde{f}_i}{\partial \tilde{H}_j \partial \tilde{H}_k}\right|_{\tilde{\mathbf{H}}=\tilde{\mathbf{H}}_\mathbf{0}}, \tag{9a}$$

$$F3_{ijkl} = \left.\frac{1}{6}\frac{\partial^3 \tilde{f}_i}{\partial \tilde{H}_j \partial \tilde{H}_k \partial \tilde{H}_l}\right|_{\tilde{\mathbf{H}}=\tilde{\mathbf{H}}_\mathbf{0}}. \tag{9b}$$

We define then $\mathbf{A}$, $\mathbf{F2}$, and $\mathbf{F3}$, respectively as tensors of size $(N_{aug} \times N_{aug})$, $(N_{aug} \times N_{aug} \times N_{aug})$, and $(N_{aug} \times N_{aug} \times N_{aug} \times N_{aug})$ corresponding to 1st, 2nd, and 3rd order terms calculated at the initial point $\tilde{\mathbf{H}}_\mathbf{0}$. Equation (8) can be expressed compactly as

$$\frac{d\tilde{\mathbf{H}}}{dt} = \mathbf{A}\tilde{\mathbf{H}} + \mathbf{\Gamma}(\tilde{\mathbf{H}}), \tag{10}$$

where $\mathbf{\Gamma}(\tilde{\mathbf{H}}) = \mathbf{F2}(\tilde{\mathbf{H}}) + \mathbf{F3}(\tilde{\mathbf{H}})$, with $\mathbf{\Gamma}(\mathbf{0}) = \mathbf{0}$. In the following subsection, the normal form of (10) is derived.

C. Normal form

Let us denote by $\mathbf{U}_i$, and $\mathbf{V}_i$ the i-th columns of right and left eigenvectors respectively, $\mathbf{U} = [u_{ij}]$, $\mathbf{V} = [v_{ij}]$, the corresponding matrices and $\mathbf{\Lambda} = \mathbf{V}^T\mathbf{A}\mathbf{U} = \mathrm{diag}(\lambda_p)$, the diagonal matrix of its eigenvalues, $p, i, j = 1, \dots N_{aug}$. It has been assumed that the matrix $\mathbf{A}$ is diagonalizable. Utilizing the transformation:

$$\tilde{\mathbf{H}} = \mathbf{U}\mathbf{y} \tag{11}$$

in (10) and multiplying the result by the left eigenvectors yields:

$$\frac{d\mathbf{y}}{dt} = \mathbf{\Lambda}\mathbf{y} + \mathbf{C2}(\mathbf{y}) + \mathbf{D3}(\mathbf{y}), \tag{12}$$

where

$$\mathbf{C2}(\mathbf{y}) + \mathbf{D3}(\mathbf{y}) = \mathbf{V}^T\mathbf{\Gamma}(\mathbf{U}\mathbf{y}),$$

or, for all $p, q, r, s = 1, 2, \ldots N_{aug}$, the p-th component is given by:

$$\begin{aligned}\frac{dy_p}{dt} = \lambda_p y_p + \sum_{q=1}^{N_{aug}}\sum_{r=1}^{N_{aug}} C_{pqr} y_q y_r + \ldots \\ \sum_{q=1}^{N_{aug}}\sum_{r=1}^{N_{aug}}\sum_{s=1}^{N_{aug}} D_{pqrs} y_q y_r y_s,\end{aligned} \tag{13}$$

where $D_{pqrs} = F3_{ijkl}v_{ip}u_{jq}u_{kr}u_{ls}$ and $C_{pqr} = F2_{ijk}v_{ip}u_{jq}u_{kr}$.

NF theory consists in transforming (12) into a system of linear or semi-linear time differential equations by a nonlinear near-identity transformation, written as [8]:

$$\mathbf{y} = \mathbf{z} + \mathbf{h2}(\mathbf{z}) + \mathbf{h3}(\mathbf{z}), \tag{14}$$

where $\mathbf{z}$ is the state variable in NF coordinate, $\mathbf{h2}$ and $\mathbf{h3}$ are respectively complex valued quadratic and cubic polynomials in $\mathbf{z}$ with $h2^j_{kl}$ and $h3^j_{pqr}$ coefficients evaluated such that (12) is simplified. From (14), the time derivative $\frac{\mathbf{dy}}{\mathbf{dt}}$ is given by:

$$\frac{d\mathbf{y}}{dt} = \frac{d\mathbf{z}}{dt}\left(I + \frac{d\mathbf{h2}(\mathbf{z})}{dz} + \frac{d\mathbf{h3}(\mathbf{z})}{dz}\right), \tag{15}$$

For $|\mathbf{z}|$ sufficiently small, $\left(\mathbf{I} + \frac{d\mathbf{h2}(\mathbf{z})}{dz} + \frac{d\mathbf{h3}(\mathbf{z})}{dz}\right)^{-1}$ is invertible and can be approximated with a power series. Therefore, substituting (15) into (12) yields

$$\frac{d\mathbf{z}}{dt} = \mathbf{\Lambda}\mathbf{z} + \hat{\mathbf{F}}(\mathbf{z}) + \mathcal{O}(|\mathbf{z}|^4) \tag{16}$$

where

$$\begin{aligned}\hat{\mathbf{F}}(\mathbf{z}) = \mathbf{C2}(\mathbf{z}) + \mathbf{D3}(\mathbf{z}) + \mathbf{\Lambda}\mathbf{h_2}(\mathbf{z}) - \frac{d\mathbf{h_2}(\mathbf{z})}{dz}\,\mathbf{\Lambda}\mathbf{z} + \ldots \\ \mathbf{\Lambda}\mathbf{h_3}(\mathbf{z}) - \frac{d\mathbf{h_3}(\mathbf{z})}{dz}\,\mathbf{\Lambda}\mathbf{z}.\end{aligned}$$

$\mathcal{O}(|\mathbf{z}|^4)$ represents terms of order higher than 3 and are neglected. In order to remove nonlinear terms from (16), the second term $\hat{\mathbf{F}}(\mathbf{z})$ on the right hand side of (16) is set to zero. Then, the coefficients of the tensors $\mathbf{h_2}(\mathbf{z})$ and $\mathbf{h_3}(\mathbf{z})$ can be determined as [8]

$$h2_{pqr} = \frac{C_{pqr}}{\lambda_q + \lambda_r - \lambda_p},\ h3_{pqrs} = \frac{D_{pqrs}}{\lambda_q + \lambda_r + \lambda_s - \lambda_p}. \tag{17}$$

It is seen that if the denominators of (17) are near zero with significant numerators, the value of the the coefficients of the tensors $\mathbf{h_2}(\mathbf{z})$ and $\mathbf{h_3}(\mathbf{z})$ (see (17)) may be very large, leading to an inconsistent transformation. The nullity of the denominators results in special relations among the complex eigenvalues, so-called resonance or near-resonance [8]. Thus, not all nonlinear terms can be annihilated from (16). The state-augmentation scheme introduces a complex conjugate eigenvalue pair due to the forcing frequency, which leads to resonance, but in our case, the resonant terms are trivial compared to the linear terms and were thus neglected. Therefore, (16) simplifies to a completely linear system:

$$\frac{d\mathbf{z}}{dt} = \mathbf{\Lambda}\mathbf{z}, \tag{18}$$

which can be solved rapidly to get a closed-form solution

$$\mathbf{z}(t) = \mathbf{z_0}e^{\mathbf{\Lambda}t}. \tag{19}$$

The system of (18) has several advantages. Because of the NF, it is much simpler than the system (8) since it has no nonlinear terms; it can be truncated to a few relevant modes for model reductions, and it leads to a more straightforward closed-form solution. From the expression of $\mathbf{z}(t)$, we can calculate $\mathbf{y}(t)$ using (14), then $\tilde{\mathbf{H}}$ from (11); finally, we obtain $\mathbf{H}(t)$ by removing $\mathbf{X}$ from $\tilde{\mathbf{H}}$. It can be noted that with the help of NF, the cubic polynomial (8) reduces to a linear problem.

IV. Application to a 1-D Diffusion Problem

We considered a lamination with a thickness $e = 1$ mm, a conductivity $\sigma = 6$ MS, and the nonlinear $B(H)$ curve depicted in Fig. 1. To account for symmetry conditions, we discretized the Ox axis with $N_x = 30$ nodes and considered only half of the lamination thickness. We solved (2) using a time-stepping method with 100 steps per period. The solution $\mathbf{H}(t)$ of (2) is considered the reference solution, denoted by $\mathbf{H_{ref}}(t)$. We solved (2) by approximating its equivalent augmented system with (i) a linear system (i.e. setting the nonlinear terms in (10) to zero and (ii) a 3rd-order polynomial (10), which is a pre-requisite for the NF approach. The obtained $\mathbf{H}(t)$ are denoted by $\mathbf{H_{ln}}(t)$ and $\mathbf{H_{poly}}(t)$, respectively. We solved (2) by the NF method, keeping the same time steps. The $\mathbf{H}(t)$ obtained is denoted by $\mathbf{H_{nf}}(t)$. To evaluate the errors between the reference and approximate solutions as a function of the maximum applied field H_{max}, we defined the relative error ϵH for a period as:

$$\epsilon H \ = \frac{|\mathbf{H_{ref}}(t) - \mathbf{H_{approx}}(t)|}{|\mathbf{H_{ref}}(t)|}, \tag{20}$$

where $\mathbf{H_{approx}}(t)$ is either $\mathbf{H_{ln}}(t)$, $\mathbf{H_{poly}}(t)$, or $\mathbf{H_{nf}}(t)$ as the case may be. We simulated three periods for the frequencies $f = 10, 100,$ and 1000 Hz. We are only interested in the steady state, which is reached after three periods. We calculated then the error only on the last period. The error evolution

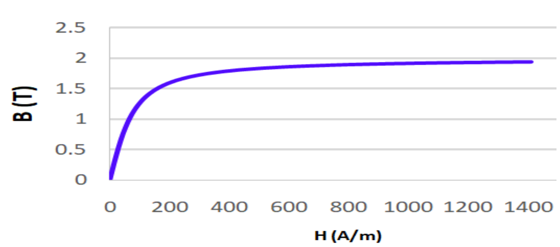

Fig. 1. $B(H)$ curve.

for the linear approximation is shown in Fig. 2. The curves show that a very narrow range of validity. If we take tolerance about 3%, it implies that the linear solution is valid only

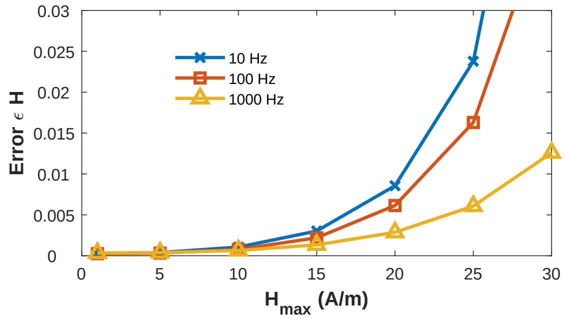

Fig. 2. Relative error for the Linear solution.

up to $H_{max} = 24$ A/m at 10 Hz and about $H_{max} = 27$ A/m at 100 Hz. The above result demonstrates that the linear system does not effectively approximate the nonlinear problem, emphasizing the importance of a higher-order approximation.

With cubic approximation, Fig. 3 points that the maximum ϵH is less than 0.15%, indicating the 3rd-order polynomial is accurate even when the lamination is saturated. In fact, even for $H_{max} = 1400\ A/m$, which corresponds to a very high level of saturation (see Fig. 1), the error is lower than 0.15% for all frequencies. The observed outcome is interesting as it suggests

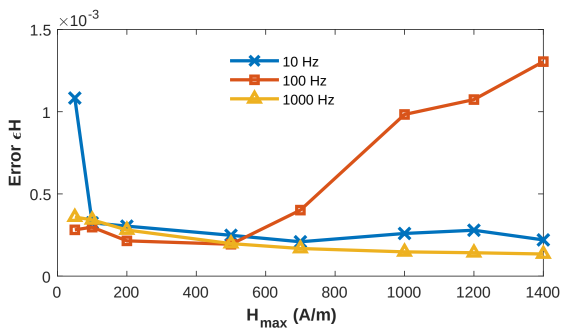

Fig. 3. Relative error for the cubic polynomial solution.

that NF application may be promising. In the NF case, Fig. 4 shows an increased range of validity in comparison with the linear case. Keeping the same error tolerance, the NF solution

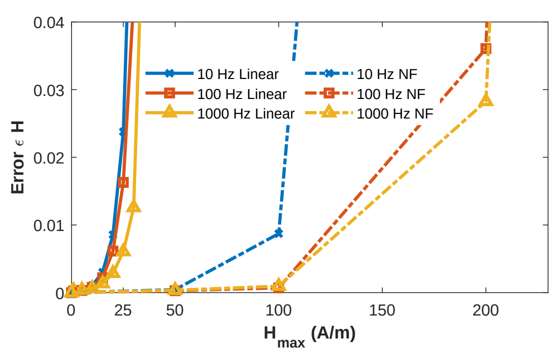

Fig. 4. Relative error for the NF solution compared to linear solution.

is valid only up to $H_{max} = 100$ A/m at 10 Hz, about $H_{max} = 200$ A/m at 100 and 1000 Hz. Though improved from the linear case, the NF result is far from that of cubic polynomial from where it was derived. The sources of this discrepancy has to be investigated in-depth. The inaccuracy could be due to the limitation of the intuitive state-augmentation method employed to convert (2) to an autonomous system. The transformation (14) is near-identity, implying that $\mathbf{y_0}$ should be near $\mathbf{z_0}$ for it to be accurate. With H_{max} as a part of the initial condition for the augmented system (see (5)), it is uncertain that the near-identity transformation will be valid if H_{max} becomes large. Moreover, the intuitive state-augmentation scheme takes the forcing frequency ω into the state matrix $\mathbf{A}$ in a form that affects its conditioning. A time-dependent NF derivation, preserving the forcing term, is not impossible and can be pursued in future to improve the solution.

V. Conclusion

In this paper, the NF method was employed to obtain a closed-form solution to a 1-D diffusion problem. The results showed that a polynomial approximation (3rd order in this study), a fundamental step for the NF procedure, is accurate. However, the NF solution has a restricted region of validity. Being the first investigation of the NF technique in this field, several future works are envisaged for amelioration. The accuracy of the cubic expansion suggests the NF analysis can be improved to obtain a wider-range valid analytical solution of the diffusion problem. For example, a time-dependent NF will be pursued via a time-dependent near-identity transformation such as the one proposed in [9]. Although only a 1-D scenario was studied, validations on 1-D problems can lead to extensions to 2-D and 3-D problems.

Acknowledgment

This work is a part of the CE2I project, co-financed by the European Union, the European Regional Development Fund (ERDF), the French State, and the Hauts-de-France Region.

References

[1] C. Touzé, O. Thomas, and A. Chaigne. Hardening/softening behaviour in non-linear oscillations of structural systemsŁ using non-linear normal modes. Journal of Sound Vibration, 273(1-2) :77–101, 2004.

[2] C. Touzé and M. Amabili, “Nonlinear normal modes for damped geometrically nonlinear systems: Application to reduced-order modelling of harmonically forced structures,” *Journal of Sound & Vibration*, vol. 298,pp. 958–981, 2006.

[3] C. Touzé, A. Vizzaccaro, and O. Thomas. Model order reduction methods for geometrically nonlinear structures : a review of nonlinear techniques. Nonlinear Dynamics, 105 :1141–1190, 2021.

[4] A. Opreni, A. Vizzaccaro, A. Frangi and C. Touzé: Model order reduction based on direct normal form: application to large finite element MEMS structures featuring internal resonance, Nonlinear Dynamics, 105, 1237-1272, 2021.

[5] T. Tian, X. Kestelyn, O. Thomas, H. Amano, and A. R. Messina. An accurate third-order normal form approximationfor power system nonlinear analysis. IEEE Transactions on Power Systems, 33(2) :2128–2139, 2018.

[6] N. S. Ugwuanyi, X. Kestelyn, O. Thomas, B. Marinescu and A. R. Messina, ”A New Fast Track to Nonlinear Modal Analysis of Power System Using Normal Form,” *IEEE Transanction on Power Systems*, vol. 35, no. 4, pp. 3247–3257, 2020.

[7] P. M. B. Waswa and S. Redkar, ”A direct approach for simplifying nonlinear systems with external periodic excitation using normal forms”, *Nonlinear Dynamics*, vol. 99, no. 2, pp. 1065–1088, 2020.

[8] A. H. Nayfeh, ”The Method of Normal Forms”, *2nd ed. Weinheim: WILEY-VCHVerlag GmbH & Co. KGaA*, 2011.

[9] A. P. Gabale and S. C. Sinha, ”A direct analysis of nonlinear systems with external periodic excitations via normal forms”, *Nonlinear Dynamics*, vol. 55, pp. 79–93, 2009.