Overview of the theoretical relations between necking and strain localization criteria

G. ALTMEYER[a], F. ABED-MERAIM[b], T. BALAN[b]

a. *Université de Lyon, CNRS, Laboratoire de Mécanique des Contacts et des Structures, UMR5259, Insa de Lyon, 18-20 rue des Sciences, F-69621 VILLEURBANNE*
b. *Arts et Métiers ParisTech, Laboratoire d'Etude des Microstructures et de Mécanique des Matériaux, UMR7239, 4 rue Fresnel, F-57078 METZ*

Résumé :

De nombreux critères de striction diffuse, de striction localisée ou de localisation sous forme de bandes ont été développés durant les dernières décennies, mais le manque de confrontation de ces modèles entre eux sur des applications pertinentes rend leur choix délicat pour le concepteur. Il est proposé de reformuler les critères d'instabilité plastique dans un cadre unifié et de comparer leurs bases théoriques afin d'établir des liens entre eux et de mettre en évidence leurs limites. Dans le cas de la striction diffuse, un rapprochement est établi entre le critère de bifurcation par point limite et le critère de force maximum dans le cas de matériaux élasto-plastiques. Pour les modes localisés, il est montré que les prédictions de l'approche de Marciniak – Kuczynski, basée sur un modèle multi-zones, tendent vers celles du critère de perte d'ellipticité lorsque la taille du défaut initial tend vers zéro. Dans le cas de comportements élasto-viscoplastiques, une approche basée sur l'analyse linéaire de stabilité est évoquée.

Abstract:

Many criteria have been developed during last decades to predict diffuse or localized necking and shear banding. The lack of confrontation of these models with each other on relevant applications makes their choice difficult for the designer. It is proposed to reformulate these plastic instability criteria in an unified framework, to compare their theoretical bases to establish links between them and then to highlighten their limitations. In the case of diffuse necking, a comparison is made between the criteria based on bifurcation analysis and on those based on maximum force principle for elastic-plastic materials. In the case of localized modes, it is shown that the predictions of the Marciniak – Kuczynski approach, based on a multi-zone model, tend to those of the loss of ellipticity criterion when the initial defect size tends to zero (no initial defect introduced). In the case of elasto-viscoplastic behavior, an approach based on a linear stability analysis is mentioned.

Keywords: necking, strain localization, plastic instability, bifurcation, maximum force criterion, Marciniak – Kuczynski method.

1 Introduction

Numerical simulation is a way to reduce the time and the costs associated to design and development of a range of deep-drawing processes. The use of such methods in design offices remains limited by the complexity of the choice of adapted couples of a material behavior model and an instability criterion. Many criteria used to predict diffuse or localized necking and shear banding have been developed during the last decades, but the lack of confrontation of these models on relevant applications makes their choice difficult for designers.

Four main approaches can be often encountered when studying these phenomena. A first approach is based on the Maximum Force principle, according to which diffuse necking is related to the maximum load during a tensile test [1-2]. This approach has later been extended to the prediction localized modes by using additional conditions [3]. Some interesting trends are found when comparing the results obtained with these criteria and experimental forming limit diagrams, but their use with advanced material modeling seems limited by necessary, and sometimes fastidious, analytical developments. To overcome these restrictions, criteria based on multi-zone approaches may be a solution. Marciniak – Kuczynski (M – K) model is based on the introduction of an initial defect, namely a band of reduced thickness, in which localized necking is supposed to occur during loading [4]. The theoretical bases of this method are however weakened by the requirement of arbitrarily user defined parameters, such as the initial defect size or the threshold value used to compare the evolution of the mechanical properties in the different zones of the metal sheet. Bifurcation analysis gives a general framework to obtain criteria with stronger theoretical bases. According to this approach, a necessary condition for diffuse necking is given by the loss of positivity of the second order work [5-6]. The loss of ellipticity criterion is developed to predict localized necking or shear banding [7]. This criterion is restricted to both rate independent materials and softening behavior. To extend the prediction of formability to viscous media, a stability analysis by a linear perturbation method may be used, necking and localization being seen as an instability of the mechanical equilibrium [8-9].

A general framework is proposed to write criteria based on these approaches, facilitating the comparison and the observation of relations existing between them.

2 Material modeling

A phenomenological modeling is considered here to represent the effects of elasticity, initial and induced anisotropy, hardening and softening. The details of its formulation are given in [10]. This modeling is based on a hypo-elastic law:

$$\dot{\boldsymbol{\sigma}} = \mathbf{C} : \left(\mathbf{D} - \mathbf{D}^p \right) \tag{1}$$

where $\mathbf{C}$ is the elastic modulus, relating the rate of the Cauchy stress tensor σ to the elastic strain rate tensor defined as the difference between the total strain rate $\mathbf{D}$ and the plastic strain rate $\mathbf{D}^p$. This tensor can be computed from the associated plastic flow law: $\mathbf{D}^p = \dot{\lambda} \dfrac{\partial f}{\partial \boldsymbol{\sigma}}$, with $\dot{\lambda}$ the plastic multiplier and f a potential that can be written under the Kuhn – Tucker form:

$$f = \bar{\sigma}\left(\boldsymbol{\sigma}, \mathbf{X}\right) - Y \leq 0 \qquad \dot{\lambda} \geq 0 \qquad \dot{\lambda} f = 0 \tag{2}$$

where $\bar{\sigma}$, Y and $\mathbf{X}$ denote respectively the equivalent stress, the size of the loading surface and the kinematic hardening variable. The chosen evolution of the kinematic hardening is represented by Armstrong – Frederick non linear law, that can be written: $\dot{\mathbf{X}} = \mathbf{H}_x \dot{\lambda}$. The current size of the loading surface is related to the initial size of the elastic domain Y_0 and to the isotropic variable R: $Y = Y_0 + R$. Different laws may be used to describe the evolution of the isotropic hardening variable, as for example Hollomon, Swift or Voce laws, relating the isotropic hardening rate to the plastic multiplier: $\dot{R} = H_R \dot{\lambda}$. Combining the previous equations, one can write the expression of the plastic multiplier and then establish the relation between the stress rate and strain rate tensors:

$$\dot{\boldsymbol{\sigma}} = \left(\mathbf{C} - \alpha^{ep} \frac{\left(\mathbf{C} : \dfrac{\partial f}{\partial \boldsymbol{\sigma}} \right) \otimes \left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{C} \right)}{\dfrac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{C} : \dfrac{\partial f}{\partial \boldsymbol{\sigma}} + \dfrac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{H}_{\mathbf{X}} + H_Y} \right) : \mathbf{D} = \mathbf{L} : \mathbf{D} \tag{3}$$

where **L** is the tangent modulus and α^{ep} a plastic load indicator that is nil during elastic loading or unloading or equals to the unity during plastic loading. This method is applicable to a wide range of material modeling and can be used to introduce other behavior, as for example softening. Different approaches have been developed to represent the effects of damage. Damage is related to the surface density of micro-defects present in an elementary volume element. Following the continuous damage mechanic framework and Lemaitre's approach, the effective stress is related to the usual stress such that:

$$\boldsymbol{\sigma}_{eff} = \frac{\boldsymbol{\sigma}}{1-d} \tag{4}$$

where d represents the isotropic damage variable. The evolution of this variable is given by: $\dot{d} = H_d \dot{\lambda}$. Adopting the strain equivalence principle and combining these new relations, one can obtain the relation between the Cauchy stress rate and the strain rate tensors:

$$\dot{\boldsymbol{\sigma}} = \left((1-d)\mathbf{C} - \alpha^{ep} \frac{\left(\mathbf{C} : \dfrac{\partial f}{\partial \boldsymbol{\sigma}_{eff}}\right) \otimes \left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}_{eff}} : \mathbf{C}\right) + H_d \boldsymbol{\sigma}_{eff} \otimes \left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}_{eff}} : \mathbf{C}\right)}{\dfrac{\partial f}{\partial \boldsymbol{\sigma}_{eff}} : \mathbf{C} : \dfrac{\partial f}{\partial \boldsymbol{\sigma}} + \dfrac{\partial f}{\partial \boldsymbol{\sigma}_{eff}} : \mathbf{H}_{\mathbf{X}} + H_Y} \right) : \mathbf{D} \tag{5}$$

It is worth noting that when damage is nil, Equations (3) and (5) defining this tensor become equivalent. A detailed development of these models is given in [10].

3 Necking and strain localization criteria

3.1 Diffuse necking

According to the bifurcation analysis approach, diffuse necking is seen as the change from a quasi-homogeneous mechanical state to a heterogeneous one. Introduced by Drucker [5] and Hill [6], General Bifurcation Criterion (GBC), proposes a lower bound for diffuse necking exclusion, defined as:

$$\frac{\partial \mathbf{V}}{\partial \mathbf{X}} : \mathcal{L} : \frac{\partial \mathbf{V}}{\partial \mathbf{X}} > 0 \tag{6}$$

A sufficient condition for uniqueness of solution of the boundary problem is the positive-definiteness of the quadratic form (6). It can also be seen as the singularity of the symmetric part of tangent modulus $\mathcal{L}$ relating the first Piola – Kirchhoff tensor and the velocity gradient $\partial \mathbf{V} / \partial \mathbf{X}$. As a particular case of general bifurcation, diffuse necking is associated with a stationary state of the nominal stress. Taking into account material behavior equations, one may obtain:

$$\mathcal{L} : \frac{\partial \mathbf{V}}{\partial \mathbf{X}} = \mathbf{0} \tag{7}$$

Limit Point Bifurcation (LPB) is associated with the singularity of the tangent modulus and is reached for the first nil eigenvalue of the tangent modulus $\mathcal{L}$.

On another side, necking criteria derived from the Maximum Force principle are based on experimental observations according to which plastic instability occurs when the force reaches its maximum during a tensile test. Extending this observation to bi-dimensional loadings, Swift introduces the formulation of the Maximum Force Criterion (MFC) [2]. Necking is then related to the maximum of both major and minor applied forces, $\dot{F}_1$ and $\dot{F}_2$ respectively:

$$\dot{F}_1 = 0 \qquad \text{and} \qquad \dot{F}_2 = 0 \tag{8}$$

The application of these hypotheses and material modeling equations lead to the classical formulation of the MFC, namely for elasto-plastic undamaged media:

$$\frac{\dot{\bar{\sigma}}}{\bar{\sigma}\,\dot{\bar{\varepsilon}}} = \frac{\sigma_1 \left(\dfrac{\partial \bar{\sigma}}{\partial \sigma_1} \right)^2 + \sigma_2 \left(\dfrac{\partial \bar{\sigma}}{\partial \sigma_2} \right)^2}{\sigma_1 \dfrac{\partial \bar{\sigma}}{\partial \sigma_1} + \sigma_2 \dfrac{\partial \bar{\sigma}}{\partial \sigma_2}} \tag{9}$$

Summarizing previous hypotheses presented during the formulation of MFC, this criterion is based on the stationarity of the applied loads at the initiation of diffuse necking and on in plane loading, i.e. $\dot{F}_3 = 0$. At this step, one can write this condition in term of nominal stresses: $\dot{\mathbf{N}} = \mathbf{0}$. When these conditions are verified in a solid, General Bifurcation condition is always satisfied. MFC can be seen as a sufficient condition for general bifurcation and GBC is more conservative than MFC. This relation is illustrated in Figure 1 with a virtual steel modeled with Voce isotropic hardening law.

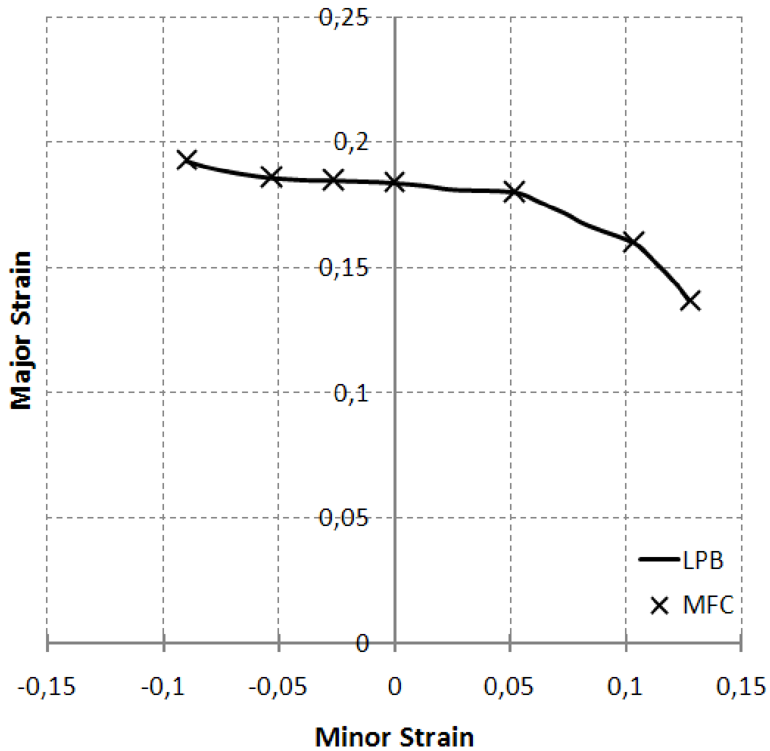

Figure 2: Forming Limit Diagram of a virtual steel obtained with Limit Point Bifurcation and Maximum Force diffuse necking criteria.

Introducing the relation between nominal stress tensor and velocity gradient, one can obtain an expression similar to Equation (7). For non trivial solution, MFC is then related to the singularity of the tangent modulus $\boldsymbol{L}$, a condition that is equivalent to the LPB condition when Swift's conditions are applied. LPB can be interpreted as a generalized form of MFC to three dimensional and non proportional loads for elastic-plastic materials [11].

3.2 Strain localization

As strain localization under shear bands and localized necking are often precursors of cracks, it is particularly important to be able to have reliable models to predict them. Loss of Ellipticity [6] and Marciniak - Kuczynski criteria [4] will be presented and compared in this part.

According to bifurcation analysis, the appearance of shear bands in a solid location is seen as a sudden evolution of the velocity gradient, from a quasi-homogeneous state to a heterogeneous state with discontinuity plane. The strain then concentrates in a localization area defined by two planes of normal **n**. Rice criterion is based on the existence of a localization band satisfying equilibrium across the band and a kinematic compatibility condition. Noting [**G**] the difference of the velocity gradients within and outside a possible localization band, the compatibility condition leads to write the discontinuity of the velocity gradient as:

$$\left[\mathbf{G}\right] = \dot{\mathbf{c}} \otimes \mathbf{n} \tag{10}$$

with $\dot{\mathbf{c}}$ the relative speed of points on either side of the localization band. Considering now the equilibrium condition across the plane, one can write:

$$\mathbf{n} \cdot \left[\dot{\mathbf{N}}\right] = \mathbf{0} \tag{11}$$

where **N** is the nominal stress tensor. Combining these equations and constitutive equations leads to the expression of Rice criterion:

$$\det\left(\mathbf{n} \cdot \boldsymbol{L} \cdot \mathbf{n}\right) = 0 \tag{12}$$

The localization is then predicted if there exists a direction of the band that cancels the determinant of the acoustic tensor. For an elasto-plastic material, softening is necessary, which justifies the use of the model coupled with damage.

Marciniak - Kuczynski criterion is based on a multi-zone method, where the evolutions of mechanical properties in different areas are compared. It predicts strain localization when the evolution of the plastic deformation is concentrated in one of them. Following M – K approach, an initial defect is introduced into the sheet in the form of a band of reduced thickness, with normal **n** at an angle θ relative to the main loading direction. The equilibrium equations through the band read:

$$f^{MK} \mathbf{n} \cdot \dot{\mathbf{N}}^{B} = \mathbf{n} \cdot \dot{\mathbf{N}} \tag{13}$$

with f^{MK} the current size of introduced defect. Behavior equations and strain compatibility conditions are used to express the relative velocity across the localization band:

$$\dot{\mathbf{c}} = \left(\mathbf{n} \cdot \boldsymbol{L}^B \cdot \mathbf{n}\right)^{-1} \cdot \mathbf{n} \cdot \left(\frac{\boldsymbol{L}}{f^{MK}} - \boldsymbol{L}^B \right) : \mathbf{G} \qquad (14)$$

Strain tensors are known inside and outside the band. The localization is then predicted when the ratio of plastic strain rate in these areas exceeds a predefined threshold. This criterion proved popular and easy to adapt to a new class of material behaviors. Its theoretical bases, however, suffer from the use of two arbitrarily defined user parameters. In the case of a linear and direct velocity gradient loadings, the orientation of the normal to the band can be analytically determined at every time as a function of the initially imposed direction $\mathbf{n}_0$. By rewriting the criterion in a lagrangian configuration, the expression of the relative velocity vector becomes:

$$\left[\dot{\mathbf{F}}\right] = \dot{\mathbf{F}}^B - \dot{\mathbf{F}} = \dot{\mathbf{c}}_0 \otimes \mathbf{n}_0 \qquad (15)$$

Reconsidering M - K criterion when the initial defect tends to be absent, the behavior inside the band tends to the behavior observed outside, then from previous equations:

$$\lim_{f_0^{MK} \to 1} \left(\mathbf{n}_0 \cdot \mathcal{L}^T \cdot \mathbf{n}_0\right) \cdot \dot{\mathbf{c}}_0 = \mathbf{0} \qquad (16)$$

At the onset of localization, relative speed vector does not tend to 0, the localization is then linked to a singularity of the pseudo acoustic tensor:

$$\det\left(\mathbf{n}_0 \cdot \mathcal{L}^T \cdot \mathbf{n}_0\right) = \mathbf{0} \qquad (17)$$

The localization is related to an increase of the relative velocity, which can be obtained when the pseudo acoustic tensor $\mathbf{n}_0 \cdot \mathcal{L}^T \cdot \mathbf{n}_0$ becomes singular, corresponding to the lagrangian form of Rice criterion. It is now shown that the predictions obtained with the M - K model tend to those of the loss of ellipticity criterion when the initial defect size tends to 0 [12]. This theoretical result is illustrated on a another virtual steel in Figure 2.

On another hand, the bifurcation analysis is restricted to time independent materials. The linear stability analysis provides a theoretical framework for the extension of these criteria to the case of elasto-viscoplastic materials. Material stability is considered using Lyapunov stability theorem for autonomous systems. A system is said stable if a small perturbation introduced in the equilibrium equations leads to small variations of the global response. Strain rate dependence may be introduced in the expression of the isotropic hardening. The strain and stress states are the obtained by the resolution of a system depending on the stress state and on the evolution of hardening and damage variables. Combining this material behavior system with the perturbed equilibrium equation one may obtain a first order differential system. Material stability is then analyzed by studying the sign of the perturbation growth rate. Application of this method when materials tend to elasto-plastic behaviors lead to results similar to those obtained with bifurcation criteria.

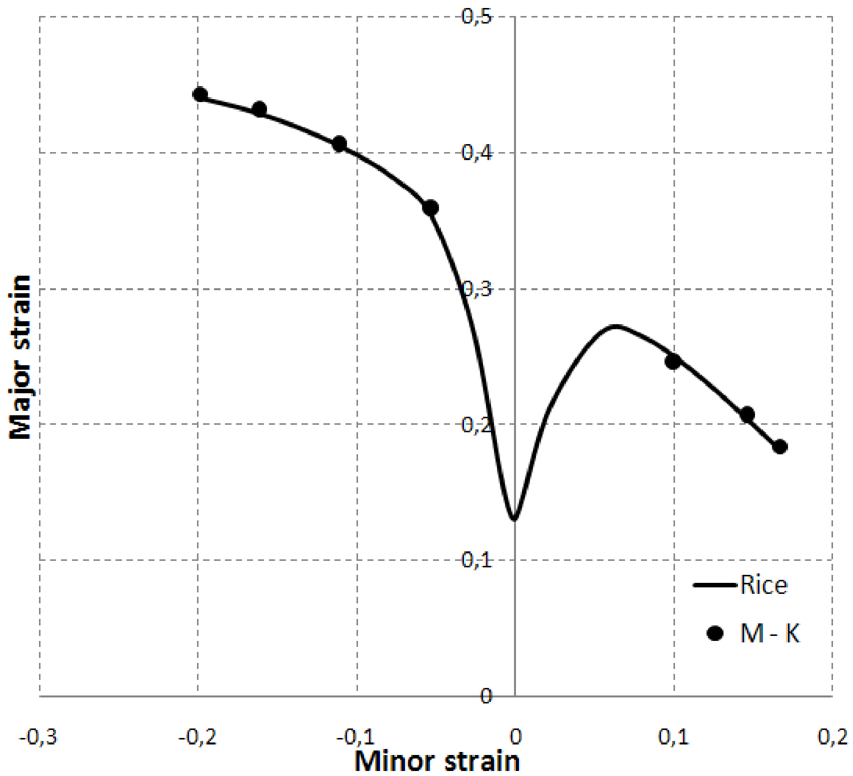

Figure 2: Forming Limit Diagram of a virtual steel obtained with Rice loss of ellipticity and M – K criteria.

4 Conclusions

Plastic instability criteria are a key for the numerical simulation of necking and localization phenomena that may occur during deep-drawing operations. The theoretical bases of the key criteria developed are compared in this paper, which can establish relationships between them. It is shown in the case of diffuse necking that limit point bifurcation and maximum force criteria are based on similar assumptions. The limit point bifurcation criterion can be seen as a generalization of the MFC. In the case of strain localization in the form of bands, it is shown that Marciniak – Kuczynski predictions tend those of Rice criterion when the size of the initial defect tends to 0. The method of linear stability analysis is finally mentioned to provide a suitable framework for the extension of material instability criteria to elasto-viscoplastic media.

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