The polar analysis of the Third-order Shear Deformation Theory of laminates

Marco Montemurro *

Arts et Métiers ParisTech, I2M CNRS UMR 5295, F-33400 Talence, France

ABSTRACT

In this paper the Verchery's polar method is extended to the conceptual framework of the Reddy's Third-order Shear Deformation Theory (TSDT) of laminates. In particular, a mathematical representation based upon tensor invariants is derived for all the laminate stiffness matrices (basic and higher-order stiffness terms). The major analytical results of the application of the polar formalism to the TSDT of laminates are the generalisation of the concept of a *quasi-homogeneous* laminate as well as the definition of some new classes of laminates. Moreover, it is proved that the elastic symmetries of the laminate shear stiffness matrices (basic and higher-order terms) depend upon those of their in-plane counterparts. As a consequence of these results a unified formulation for the problem of designing the laminate elastic symmetries in the context of the TSDT is proposed. The optimum solutions are found within the framework of the polar-genetic approach, since the objective function is written in terms of the laminate polar parameters, while a genetic algorithm is used as a numerical tool for the solution search. In order to support the theoretical results, and also to prove the effectiveness of the proposed approach, some new and meaningful numerical examples are discussed in the paper.

1. Introduction

As well known, the Classic Laminate Theory (CLT) together with the First-order Shear Deformation Theory (FSDT) are the simplest theories employed for describing the mechanical behaviour of a composite laminate considered as an equivalent homogeneous (generally) anisotropic plate. Such theories properly describe the laminate kinematic response in the case of small (CLT) or moderate (FSDT) values of the plate characteristic aspect ratio (i.e. the ratio of its thickness to its shorter side). However, the major drawback of these theories is in the estimation of the influence of the laminate transverse shear stiffness on its mechanical response (which becomes more and more important for thick plates). On one hand, in the case of the CLT the laminate transverse shear stiffness does not intervene in the definition of the laminate constitutive equation (making this theory adequate only for thin laminates). On the other hand, in the framework of the FSDT the influence of the transverse shear stiffness is taken into account within the definition of the laminate constitutive behaviour. Nevertheless, due to the kinematic model on which the FSDT relies, the

through-the-thickness shear stresses are constant within each constitutive layer, leading in this way to a mechanical contradiction. Indeed, the shear stresses do not satisfy: (a) the boundary conditions on the external faces of the laminate, (b) the local equilibrium equations (elasticity solution) and (c) the continuity condition at the layers interface, see [1]. To overcome these contradictions, it is common to introduce the so-called "shear correction factor" [1,2] which generally satisfies only two of the previous three conditions. However, in the context of the FSDT, the definition of the shear correction factor is immediate only for isotropic plates, while it becomes more arduous defining such a quantity for a laminate since it depends upon the geometrical parameters of the stack (layers orientations and positions) [1].

Higher order theories allow for overcoming such a difficulty: they give a better description of both the laminate kinematics and stress field without the need of introducing any correction coefficient. However these theories require the introduction of higher-order stress resultants and stiffness matrices whose physical meaning is not immediate. In literature one can find several higher-order theories of different nature: for each theory the displacement field is expanded in a finite series (in terms of the thickness coordinate) of unknown functions: the terms of the series (i.e. the functions depending upon the thickness coordinate) can belong to a given basis (polynomial, trigonometric, radial, B-spline,

* Tel.: +33 55 68 45 422; fax: +33 54 00 06 964.
E-mail addresses: marco.montemurro@ensam.eu, marco.montemurro@u-bordeaux1.fr

Notations

CLT classical laminate theory
FSDT First-order Shear Deformation Theory
TSDT Third-order Shear Deformation Theory
GA genetic algorithm
$\Gamma = \{O; x_1, x_2, x_3\}$ local (or material) frame of the elementary ply
$\Gamma^l = \{O; x, y, z = x_3\}$ global frame of the laminate
θ rotation angle
$\{11, 22, 33, 32, 31, 21\} \Longleftrightarrow \{1, 2, 3, 4, 5, 6\}$ correspondence between tensor and Voigt's (matrix) notation for the indexes of tensors (local frame)
$\{xx, yy, zz, zy, zx, yx\} \Longleftrightarrow \{x, y, z, q, r, s\}$ correspondence between tensor and Voigt's (matrix) notation for the indexes of tensors (global frame)
$Z_{ij}, \ (i, j = 1, 2 \text{ or } i, j = x, y)$ second-rank plane tensor using tensor notation (local and global frame)
$L_{ijkl}, \ (i, j, k, l = 1, 2 \text{ or } i, j, k, l = x, y)$ fourth-rank plane tensor using tensor notation (local and global frame)
u, v, w components of the laminate displacement field within the global frame Γ^l
$u_0, v_0, w_0, \phi_x, \phi_y$ the five independent kinematic unknowns in the context of the Reddy's TSDT
n number of layers
$\{\delta_k\} \ (k = 1, \ldots, n)$ vector of the layers orientation angles
z_{k-1}, z_k thickness coordinates of bottom and top faces of the kth constitutive ply, respectively
h overall thickness of the laminate
$\{\varepsilon^{(0)}\}, \{\varepsilon^{(1)}\}, \{\varepsilon^{(3)}\}, 3 \times 1$ vectors of in-plane strains of the laminate middle plane
$\{\gamma^{(0)}\}, \{\gamma^{(2)}\}, 2 \times 1$ vectors of the transverse shear strains of the laminate middle plane
$\{N\}, \{M\}, \{P\}, 3 \times 1$ vectors of higher-order generalised in-plane forces (per unit length)
$\{Q\}, \{R\}, 2 \times 1$ vectors of higher-order generalised transverse shear forces (per unit length)
$[Q], 3 \times 3$ in-plane reduced stiffness matrix of the constitutive lamina
$[\widehat{Q}], 2 \times 2$ out-of-plane reduced stiffness matrix of the constitutive lamina
$T_0, T_1, R_0, R_1, \Phi_0, \Phi_1$ polar parameters of a fourth-rank plane tensor (also used for the lamina in-plane reduced stiffness matrix $[Q]$)

T, R, Φ polar parameters of a second-rank plane tensor (also used for the lamina transverse shear reduced stiffness matrix $[\widehat{Q}]$)
$[A], [B], [D], [E], [F], [H], 3 \times 3$ in-plane stiffness matrices of the laminate (membrane, membrane/bending coupling, bending and higher-order stiffness, respectively)
$[A^*], [B^*], [D^*], [E^*], [F^*], [H^*], 3 \times 3$ homogenised in-plane stiffness matrices of the laminate (membrane, membrane/bending coupling, bending and higher-order stiffness, respectively)
$[\widehat{A}], [\widehat{D}], [\widehat{F}], 2 \times 2$ transverse shear stiffness matrices of the laminate (basic and higher-order stiffness, respectively)
$[\widehat{A}^*], [\widehat{D}^*], [\widehat{F}^*], 2 \times 2$ homogenised transverse shear stiffness matrices of the laminate (basic and higher-order stiffness, respectively)
b_k, d_k, e_k, f_k, h_k coefficients of the laminate stiffness matrices
$T_{0M^*}, T_{1M^*}, R_{0M^*}, R_{1M^*}, \Phi_{0M^*}, \Phi_{1M^*}$ polar parameters of the generic homogenised in-plane stiffness matrix of the laminate $(M^* = A^*, B^*, D^*, E^*, F^*, H^*)$
$T_{\widehat{M}^*}, R_{\widehat{M}^*}, \Phi_{\widehat{M}^*}$ polar parameters of the generic homogenised transverse shear stiffness matrix of the laminate $(\widehat{M}^* = \widehat{A}^*, \widehat{D}^*, \widehat{F}^*)$
$[C_1^*], [C_2^*], [C_3^*], 3 \times 3$ laminate homogeneity matrices
$E_i, \ (i = 1, 2, 3)$ Young's moduli of the constitutive lamina (material frame)
$G_{ij}, \ (i, j = 1, 2, 3)$ shear moduli of the constitutive lamina (material frame)
$\nu_{ij}, \ (i, j = 1, 2, 3)$ Poisson's ratios of the constitutive lamina (material frame)
t_{ply} thickness of the constitutive lamina
Ψ overall objective function for the problem of designing the elastic symmetries of the laminate
$\{f\}, 37 \times 1$ vector of partial objective functions
$[W], 37 \times 37$ positive semi-definite diagonal weight matrix
$\widehat{R}_{0M^*}, \widehat{R}_{1M^*}, \widehat{\Phi}_{0M^*}, \widehat{\Phi}_{1M^*}$ imposed values for the polar parameters of matrix $[M^*], (M^* = A^*, D^*, F^*, H^*)$
N_{pop} number of populations
N_{ind} number of individuals
N_{gen} number of generations
p_{cross} crossover probability
p_{mut} mutation probability

NURBS, etc.). In principle it is possible to expand the displacement field up to any degree in terms of the thickness coordinate. Nevertheless, an expansion up to the third order (the so-called third-order theory) is sufficient to capture the quadratic variation of the transverse shear strains and stresses within each layer. There are a lot of papers on third-order theories, see for instance [3–11]. Despite they seem to differ from each other, the displacement fields of these theories are mechanically equivalent (or related), see [12]. Recently, the classical Third-order Shear Deformation Theory (TSDT) of laminates, initially introduced by Reddy [3], has been extended and reformulated according to the Eringen's nonlocal linear elasticity theory to capture small scale size effects through the thickness [13].

The aim of this paper does not consist in a critical analysis of all the different types of TSDT that can be found in literature, rather it aims to shed some light on certain aspects linked to the formulation of the laminate constitutive equation in the conceptual framework of the classical TSDT of Reddy [1]. Particularly, the objective of the present work is twofold: on one hand it aims of clarifying the physical meaning of the higher-order stiffness matrices while on the other hand it intends of estimating their influence on the

elastic response of the laminate. To these purposes the polar method initially introduced by Verchery [14], later enriched and deeply investigated by Vannucci and his co-workers [15–19] and recently extended to the FSDT of laminates [20] is here employed (for the first time) within the framework of the TSDT. In particular, the expression of the polar parameters of the laminate higher-order stiffness matrices is analytically derived. Thanks to the polar formalism and its application to the TSDT it is possible to introduce some new classes of laminates and also to generalise the definition of a *quasi-homogeneous* laminate, initially introduced by Vannucci and Verchery [21]. Accordingly, it is possible to carry out a more general analysis of the elastic response of the laminate by reformulating and generalising the problem of designing its elastic symmetries (initially introduced by Vannucci [22] and later extended to the FSDT [20]) within the context of the TSDT. This problem is formulated as an unconstrained minimisation problem in the space of the full set of the laminate polar parameters (even including the higher-order stiffness matrices). Due to its particular nature (i.e. a non-convex optimisation problem in the space of the layers orientation angles), the solution search process is performed by using the genetic algorithm (GA) BIANCA [23–25]. Finally, in

order to numerically prove and support the major analytical results found in this work, some meaningful and non-conventional examples are presented.

The paper is organised as follows: Section 2 briefly recalls the fundamentals of the polar formalism. In Section 3 the polar method is applied in the framework of the TSDT, by highlighting the major analytical results. Section 4 presents the mathematical formulation of the problem of designing the elastic symmetries of a laminate as an optimisation problem and the generalisation of this formulation when considering the laminate behaviour in the context of the TSDT. Section 5 shows some numerical results in order to prove the effectiveness of the polar formalism when it is applied to the TSDT. Finally Section 6 ends the paper with some concluding remarks.

2. Fundamentals of the polar method

For the sake of synthesis in this section the main results of the polar method introduced by Verchery in 1979 [14] are briefly recalled. The polar method is essentially a mathematical technique that allows for expressing any n-rank plane tensor through a set of tensor invariants. As a consequence, such a representation can be applied not only to elasticity-like tensors but also to a very general (even asymmetric) plane tensor, see for instance [26]. For more details on the polar formalism the reader is addressed to [15].

In the framework of the polar formalism a second-rank (symmetric) tensor Z_{ij}, $(i, j = 1, 2)$, within the local frame Γ, can be stated as:

$$
\begin{aligned}
Z_{11} &= T + R\cos 2\Phi, \\
Z_{12} &= R\sin 2\Phi, \\
Z_{22} &= T - R\cos 2\Phi,
\end{aligned}
\tag{1}
$$

where T is the isotropic modulus, R the deviatoric one and Φ the polar angle. Furthermore, for a second-rank plane symmetric tensor there are only two tensor invariants, i.e. the polar moduli T and R, while the polar angle Φ can be arbitrarily chosen to fix the reference frame. The converse relations are:

$$
\begin{aligned}
T &= \frac{Z_{11} + Z_{22}}{2}, \\
Re^{i2\Phi} &= \frac{Z_{11} - Z_{22}}{2} + iZ_{12},
\end{aligned}
\tag{2}
$$

where $i = \sqrt{-1}$ is the imaginary unit. For a second-rank plane tensor the only possible symmetry is the isotropy which can be obtained when the deviatoric modulus of the tensor is null, i.e. $R = 0$. Moreover, when using the polar formalism, the components of the second-rank tensor can be expressed in a very straightforward manner in the frame Γ^{I} (turned counter-clock wise by an angle θ around the x_3 axis) as follows:

$$
\begin{aligned}
Z_{xx} &= T + R\cos 2(\Phi - \theta), \\
Z_{xy} &= R\sin 2(\Phi - \theta), \\
Z_{yy} &= T - R\cos 2(\Phi - \theta).
\end{aligned}
\tag{3}
$$

Concerning a fourth-rank elasticity-like plane tensor L_{ijkl}, $(i, j, k, l = 1, 2)$ (expressed within the local frame Γ), its polar representation writes:

$$
\begin{aligned}
L_{1111} &= T_0 + 2T_1 + R_0\cos 4\Phi_0 + 4R_1\cos 2\Phi_1, \\
L_{1122} &= -T_0 + 2T_1 - R_0\cos 4\Phi_0, \\
L_{1112} &= R_0\sin 4\Phi_0 + 2R_1\sin 2\Phi_1, \\
L_{2222} &= T_0 + 2T_1 + R_0\cos 4\Phi_0 - 4R_1\cos 2\Phi_1, \\
L_{2212} &= -R_0\sin 4\Phi_0 + 2R_1\sin 2\Phi_1, \\
L_{1212} &= T_0 - R_0\cos 4\Phi_0.
\end{aligned}
\tag{4}
$$

As it clearly appears from Eq. (4) the six independent Cartesian components of L_{ijkl} are expressed in terms of six polar parameters: T_0 and T_1 are the isotropic moduli, R_0 and R_1 are the anisotropic ones, while Φ_0 and Φ_1 are the polar angles. Only five quantities are tensor invariants, namely the polar moduli T_0, T_1, R_0, R_1 together with the angular difference $\Phi_0 - \Phi_1$. One of the two polar angles, Φ_0 or Φ_1, can be arbitrarily chosen to fix the reference frame. The converse relations can be stated as:

$$
\begin{aligned}
8T_0 &= L_{1111} - 2L_{1122} + 4L_{1212} + L_{2222}, \\
8T_1 &= L_{1111} + 2L_{1122} + L_{2222}, \\
8R_0e^{i4\Phi_0} &= L_{1111} - 2L_{1122} - 4L_{1212} + L_{2222} + 4i(L_{1112} - L_{2212}), \\
8R_1e^{i2\Phi_1} &= L_{1111} - L_{2222} + 2i(L_{1112} + L_{2212}).
\end{aligned}
\tag{5}
$$

The Cartesian components of the fourth-rank tensor expressed in the frame Γ^{I} are:

$$
\begin{aligned}
L_{xxxx} &= T_0 + 2T_1 + R_0\cos 4(\Phi_0 - \theta) + 4R_1\cos 2(\Phi_1 - \theta), \\
L_{xxyy} &= -T_0 + 2T_1 - R_0\cos 4(\Phi_0 - \theta), \\
L_{xxxy} &= R_0\sin 4(\Phi_0 - \theta) + 2R_1\sin 2(\Phi_1 - \theta), \\
L_{yyyy} &= T_0 + 2T_1 + R_0\cos 4(\Phi_0 - \theta) - 4R_1\cos 2(\Phi_1 - \theta), \\
L_{yyxy} &= -R_0\sin 4(\Phi_0 - \theta) + 2R_1\sin 2(\Phi_1 - \theta), \\
L_{xyxy} &= T_0 - R_0\cos 4(\Phi_0 - \theta).
\end{aligned}
\tag{6}
$$

Finally, it can be proved that in the case of a fourth-rank elasticity-like plane tensor four different types of elastic symmetry exist: (a) *Ordinary orthotropy* $(\Phi_0 - \Phi_1 = K\frac{\pi}{4}, K = 0, 1)$; (b) R_0-*Orthotropy* $(R_0 = 0)$; (c) *Square symmetry* $(R_1 = 0)$ and (d) *Isotropy* $(R_0 = R_1 = 0)$. For a deeper insight in the matter the reader is addressed to [15,20].

3. Application of the polar formalism to the Third-order Shear Deformation Theory of laminates

3.1. Constitutive equations of the laminate

For the sake of simplicity in this section all of the equations governing the laminate mechanical response will be formulated in the context of the Voigt's (matrix) notation. The passage from tensor notation to the Voigt's one can be easily expressed through the well-known two-way relationships among indexes (for both local and global frames) as follows:

$$
\begin{aligned}
\{11, 22, 33, 32, 31, 21\} &\Longleftrightarrow \{1, 2, 3, 4, 5, 6\}, \\
\{xx, yy, zz, zy, zx, yx\} &\Longleftrightarrow \{x, y, z, q, r, s\}.
\end{aligned}
\tag{7}
$$

Let us consider a multilayer plate composed of n plies. Let be δ_k, z_{k-1} and z_k the orientation angle and the thickness coordinates of the bottom and top surfaces of the kth elementary lamina, respectively, as illustrated in Fig. 1. Within the conceptual framework of the classical TSDT introduced by Reddy [1] the displacement field of the laminated plate can be expressed (within the laminate global frame Γ^{I}) as:
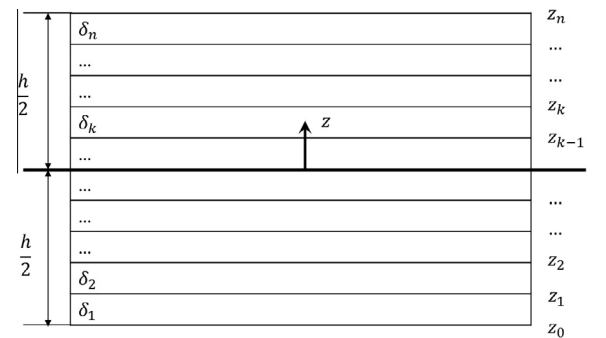

Fig. 1. Definition of the geometrical parameters of the laminate.

$$u(x,y,z) = u_0(x,y) + \phi_x(x,y)\left(z - \frac{4z^3}{3h^2}\right) - \frac{4z^3}{3h^2}\frac{\partial w_0}{\partial x},$$
$$v(x,y,z) = v_0(x,y) + \phi_y(x,y)\left(z - \frac{4z^3}{3h^2}\right) - \frac{4z^3}{3h^2}\frac{\partial w_0}{\partial y}, \tag{8}$$
$$w(x,y,z) = w_0(x,y),$$

where h is the total thickness of the laminate. In Eq. (8) u_0, v_0, w_0, ϕ_x and ϕ_y are the five independent kinematics unknowns. It is noteworthy that this displacement field engenders a transverse shear stress field having a quadratic variation through the thickness of each constitutive lamina which precisely meets the traction-free boundary conditions on the top and bottom faces of the laminate, see [1]. Taking into account such considerations, the laminate constitutive equation, under the hypothesis of small strains and moderate rotations [1], can be stated as:

$$\left\{\begin{array}{c}\{N\}\\\{M\}\\\{P\}\end{array}\right\} = \begin{bmatrix}[A] & [B] & [E]\\ [B] & [D] & [F]\\ [E] & [F] & [H]\end{bmatrix}\left\{\begin{array}{c}\{\varepsilon^{(0)}\}\\\{\varepsilon^{(1)}\}\\\{\varepsilon^{(3)}\}\end{array}\right\}, \tag{9}$$

$$\left\{\begin{array}{c}\{Q\}\\\{R\}\end{array}\right\} = \begin{bmatrix}[\widehat{A}] & [\widehat{D}]\\ [\widehat{D}] & [\widehat{F}]\end{bmatrix}\left\{\begin{array}{c}\{\gamma^{(0)}\}\\\{\gamma^{(2)}\}\end{array}\right\}. \tag{10}$$

In Eqs. (9) and (10) $\{\varepsilon^{(0)}\}, \{\varepsilon^{(1)}\}$ and $\{\varepsilon^{(3)}\}$ represent the in-plane strains of the laminate middle plane while $\{\gamma^{(0)}\}$ and $\{\gamma^{(2)}\}$ are the transverse shear strains: all of these quantities can be derived by means of the non-linear strain–displacement relationship in the case of moderate rotations and small strains. $\{N\}, \{M\}$ and $\{P\}$ are the higher-order generalised in-plane stress resultants while $\{Q\}$ and $\{R\}$ are the higher-order generalised transverse shear stress resultants. For a deeper insight in the matter (mainly about the definition of these quantities) the reader is addressed to [1]. Here the major concern is the analysis of the elastic response of the laminate through an investigation of the elastic symmetries of its higher-order stiffness characteristics. In particular, in Eqs. (9) and (10) $[A], [B], [D], [E], [F]$ and $[H]$ are the in-plane stiffness matrices (membrane, membrane/bending coupling, bending and higher-order stiffness terms) while $[\widehat{A}], [\widehat{D}]$ and $[\widehat{F}]$ are the transverse shear stiffness matrices (basic and higher-order terms) of the laminate. Such matrices are defined as follows:

$$[A] = \sum_{k=1}^{n}[Q(\delta_k)](z_k - z_{k-1}), \quad [B] = \frac{1}{2}\sum_{k=1}^{n}[Q(\delta_k)](z_k^2 - z_{k-1}^2),$$

$$[D] = \frac{1}{3}\sum_{k=1}^{n}[Q(\delta_k)](z_k^3 - z_{k-1}^3), \quad [E] = \frac{1}{4}\sum_{k=1}^{n}[Q(\delta_k)](z_k^4 - z_{k-1}^4), \tag{11}$$

$$[F] = \frac{1}{5}\sum_{k=1}^{n}[Q(\delta_k)](z_k^5 - z_{k-1}^5), \quad [H] = \frac{1}{7}\sum_{k=1}^{n}[Q(\delta_k)](z_k^7 - z_{k-1}^7),$$

$$[\widehat{A}] = \sum_{k=1}^{n}[\widehat{Q}(\delta_k)](z_k - z_{k-1}), \quad [\widehat{D}] = \frac{1}{3}\sum_{k=1}^{n}[\widehat{Q}(\delta_k)](z_k^3 - z_{k-1}^3),$$

$$[\widehat{F}] = \frac{1}{5}\sum_{k=1}^{n}[\widehat{Q}(\delta_k)](z_k^5 - z_{k-1}^5). \tag{12}$$

In the previous equations $[Q(\delta_k)]$ and $[\widehat{Q}(\delta_k)]$ are the in-plane and the transverse shear reduced stiffness matrices of the kth ply, respectively. Eqs. (11) and (12) can be simplified for a laminate composed of identical layers, i.e. layers having same material properties and thickness, as follows:

$$[A] = \frac{h}{n}\sum_{k=1}^{n}[Q(\delta_k)], \quad [B] = \frac{1}{2}\left(\frac{h}{n}\right)^2\sum_{k=1}^{n}b_k[Q(\delta_k)],$$

$$[D] = \frac{1}{12}\left(\frac{h}{n}\right)^3\sum_{k=1}^{n}d_k[Q(\delta_k)], \quad [E] = \frac{1}{8}\left(\frac{h}{n}\right)^4\sum_{k=1}^{n}e_k[Q(\delta_k)], \tag{13}$$

$$[F] = \frac{1}{80}\left(\frac{h}{n}\right)^5\sum_{k=1}^{n}f_k[Q(\delta_k)], \quad [H] = \frac{1}{448}\left(\frac{h}{n}\right)^7\sum_{k=1}^{n}h_k[Q(\delta_k)],$$

$$[\widehat{A}] = \frac{h}{n}\sum_{k=1}^{n}[\widehat{Q}(\delta_k)], \quad [\widehat{D}] = \frac{1}{12}\left(\frac{h}{n}\right)^3\sum_{k=1}^{n}d_k[\widehat{Q}(\delta_k)],$$

$$[\widehat{F}] = \frac{1}{80}\left(\frac{h}{n}\right)^5\sum_{k=1}^{n}f_k[\widehat{Q}(\delta_k)], \tag{14}$$

where the expression of the coefficients b_k, d_k, e_k, f_k and h_k and the related geometrical properties are discussed in Appendix A.

Let us consider the reduced stiffness matrices of the elementary ply. It can be easily noticed that after a change of frame (i.e. when passing from the lamina local frame Γ to the global one Γ^l) the components of matrix $[Q(\delta_k)]$ behave like those of a fourth-rank elasticity-like plane tensor, while those of $[\widehat{Q}(\delta_k)]$ vary according to the transformation law of a second-rank symmetric plane tensor, see [20]. Therefore these matrices can be represented by means of the polar formalism:

$$Q_{xx} = T_0 + 2T_1 + R_0\cos 4(\Phi_0 + \delta_k) + 4R_1\cos 2(\Phi_1 + \delta_k),$$
$$Q_{xy} = -T_0 + 2T_1 - R_0\cos 4(\Phi_0 + \delta_k),$$
$$Q_{xs} = R_0\sin 4(\Phi_0 + \delta_k) + 2R_1\sin 2(\Phi_1 + \delta_k),$$
$$Q_{yy} = T_0 + 2T_1 + R_0\cos 4(\Phi_0 + \delta_k) - 4R_1\cos 2(\Phi_1 + \delta_k), \tag{15}$$
$$Q_{ys} = -R_0\sin 4(\Phi_0 + \delta_k) + 2R_1\sin 2(\Phi_1 + \delta_k),$$
$$Q_{ss} = T_0 - R_0\cos 4(\Phi_0 + \delta_k),$$

$$\widehat{Q}_{qq} = T + R\cos 2(\Phi - \delta_k),$$
$$\widehat{Q}_{qr} = R\sin 2(\Phi - \delta_k), \tag{16}$$
$$\widehat{Q}_{rr} = T - R\cos 2(\Phi - \delta_k).$$

To be remarked that in the previous equations it is the material frame of the kth lamina (and not the global one) which is turned counter-clock wise by an angle δ_k around the x_3 axis. On the other hand, it must be pointed out that in the previous equations, the components of the out-of-plane stiffness matrix of the ply behave like those of a second-rank symmetric plane tensor with the local frame turned clockwise by an angle δ_k around the x_3 axis. In Eqs. (15) and (16) $T_0, T_1, R_0, R_1, \Phi_0$ and Φ_1 are the polar parameters of the in-plane reduced stiffness matrix of the lamina, while T, R, and Φ are those of the transverse shear stiffness matrix: all of these parameters solely depend upon the ply material properties (e.g. if the ply is orthotropic the polar parameters of $[Q(\delta_k)]$ depend upon E_1, E_2, G_{12} and v_{12}, while those of $[\widehat{Q}(\delta_k)]$ depend upon G_{23} and G_{13}).

In order to better analyse and understand the mechanical response of the laminate it is useful to homogenise the units of its characteristic stiffness matrices to those of the ply reduced stiffness matrices:

$$[A^*] = \frac{1}{h}[A], \quad [B^*] = \frac{2}{h^2}[B], \quad [D^*] = \frac{12}{h^3}[D],$$

$$[E^*] = \frac{8}{h^4}[E], \quad [F^*] = \frac{80}{h^5}[F], \quad [H^*] = \frac{448}{h^7}[H], \tag{17}$$

$$[\widehat{A}^*] = \frac{1}{h}[\widehat{A}], \quad [\widehat{D}^*] = \frac{12}{h^3}[\widehat{D}], \quad [\widehat{F}^*] = \frac{80}{h^5}[\widehat{F}].$$

In the framework of the polar formalism it is possible to express all of the previous matrices in terms of their polar parameters. In particular, matrices $[A^*], [B^*], [D^*], [E^*], [F^*]$ and $[H^*]$ behave like a fourth-rank elasticity-like plane tensor while matrices $[\widehat{A}^*], [\widehat{D}^*]$ and $[\widehat{F}^*]$ behave like a second-rank symmetric plane tensor. Moreover, the polar parameters of these matrices can be expressed as functions of the polar parameters of the lamina reduced stiffness matrices and of the geometrical properties of the stack (i.e. the layer orientation and position). The polar representation of the homogenised stiffness matrices of the laminate can be stated as:

$$T_{0A^*} = T_0,$$
$$T_{1A^*} = T_1,$$
$$R_{0A^*}e^{i4\Phi_{0A^*}} = \frac{1}{n}R_0 e^{i4\Phi_0}\sum_{k=1}^{n}e^{i4\delta_k},$$
$$R_{1A^*}e^{i2\Phi_{1A^*}} = \frac{1}{n}R_1 e^{i2\Phi_1}\sum_{k=1}^{n}e^{i2\delta_k}, \tag{18}$$

$$T_{0B^*} = 0,$$
$$T_{1B^*} = 0,$$
$$R_{0B^*}e^{i4\Phi_{0B^*}} = \frac{1}{n^2}R_0 e^{i4\Phi_0}\sum_{k=1}^{n}b_k e^{i4\delta_k},$$
$$R_{1B^*}e^{i2\Phi_{1B^*}} = \frac{1}{n^2}R_1 e^{i2\Phi_1}\sum_{k=1}^{n}b_k e^{i2\delta_k}, \tag{19}$$

$$T_{0D^*} = T_0,$$
$$T_{1D^*} = T_1,$$
$$R_{0D^*}e^{i4\Phi_{0D^*}} = \frac{1}{n^3}R_0 e^{i4\Phi_0}\sum_{k=1}^{n}d_k e^{i4\delta_k},$$
$$R_{1D^*}e^{i2\Phi_{1D^*}} = \frac{1}{n^3}R_1 e^{i2\Phi_1}\sum_{k=1}^{n}d_k e^{i2\delta_k}, \tag{20}$$

$$T_{0E^*} = 0,$$
$$T_{1E^*} = 0,$$
$$R_{0E^*}e^{i4\Phi_{0E^*}} = \frac{1}{n^4}R_0 e^{i4\Phi_0}\sum_{k=1}^{n}e_k e^{i4\delta_k},$$
$$R_{1E^*}e^{i2\Phi_{1E^*}} = \frac{1}{n^4}R_1 e^{i2\Phi_1}\sum_{k=1}^{n}e_k e^{i2\delta_k}, \tag{21}$$

$$T_{0F^*} = T_0,$$
$$T_{1F^*} = T_1,$$
$$R_{0F^*}e^{i4\Phi_{0F^*}} = \frac{1}{n^5}R_0 e^{i4\Phi_0}\sum_{k=1}^{n}f_k e^{i4\delta_k},$$
$$R_{1F^*}e^{i2\Phi_{1F^*}} = \frac{1}{n^5}R_1 e^{i2\Phi_1}\sum_{k=1}^{n}f_k e^{i2\delta_k}, \tag{22}$$

$$T_{0H^*} = T_0,$$
$$T_{1H^*} = T_1,$$
$$R_{0H^*}e^{i4\Phi_{0H^*}} = \frac{1}{n^7}R_0 e^{i4\Phi_0}\sum_{k=1}^{n}h_k e^{i4\delta_k},$$
$$R_{1H^*}e^{i2\Phi_{1H^*}} = \frac{1}{n^7}R_1 e^{i2\Phi_1}\sum_{k=1}^{n}h_k e^{i2\delta_k}, \tag{23}$$

$$T_{\widehat{A}^*} = T,$$
$$R_{\widehat{A}^*}e^{i2\Phi_{\widehat{A}^*}} = \frac{1}{n}Re^{i2\Phi}\sum_{k=1}^{n}e^{-i2\delta_k}, \tag{24}$$

$$T_{\widehat{D}^*} = T,$$
$$R_{\widehat{D}^*}e^{i2\Phi_{\widehat{D}^*}} = \frac{1}{n^3}Re^{i2\Phi}\sum_{k=1}^{n}d_k e^{-i2\delta_k}, \tag{25}$$

$$T_{\widehat{F}^*} = T,$$
$$R_{\widehat{F}^*}e^{i2\Phi_{\widehat{F}^*}} = \frac{1}{n^5}Re^{i2\Phi}\sum_{k=1}^{n}f_k e^{-i2\delta_k}. \tag{26}$$

The above equations have been derived by following the same logical procedure used in the case of the polar analysis of the FSDT, see [20]. After a quick glance to Eqs. (18)–(26) the following aspects can be highlighted:

- the isotropic polar moduli of matrices [A*], [D*], [F*] and [H*] are equal to those of the in-plane reduced stiffness matrix of the elementary ply;
- the isotropic part of matrices [B*] and [E*] is null;
- the isotropic modulus of matrices [$\widehat{A}^*$], [$\widehat{D}^*$] and [$\widehat{F}^*$] is equal to that of the transverse shear stiffness matrix of the constitutive lamina;

Moreover (as in the case of the definition of the laminate homogenised transverse shear stiffness matrix in the context of the FSDT) it can be proved that both the deviatoric modulus and the polar angle of matrices [$\widehat{A}^*$], [$\widehat{D}^*$] and [$\widehat{F}^*$] can be expressed as a linear combination of the anisotropic polar modulus R_{1M^*} and the related polar angle Φ_{1M^*} of matrices [A*], [D*] and [F*], respectively (see [20] for the details on the mathematical passages):

$$R_{\widehat{A}^*}e^{i2\Phi_{\widehat{A}^*}} = \frac{R}{R_1}R_{1A^*}e^{i2\left(\Phi+\Phi_1-\Phi_{1A^*}\right)},$$
$$R_{\widehat{D}^*}e^{i2\Phi_{\widehat{D}^*}} = \frac{R}{R_1}R_{1D^*}e^{i2\left(\Phi+\Phi_1-\Phi_{1D^*}\right)}, \tag{27}$$
$$R_{\widehat{F}^*}e^{i2\Phi_{\widehat{F}^*}} = \frac{R}{R_1}R_{1F^*}e^{i2\left(\Phi+\Phi_1-\Phi_{1F^*}\right)}.$$

Eq. (27) means that, when the material of the elementary ply is fixed a priori, the overall elastic response of the laminate depends only on the anisotropic part of matrices [A*], [B*], [D*], [E*], [F*] and [H*]. In particular, the designer can act, through a variation of the geometric parameters of the stack, only on the anisotropic polar moduli and polar angles of the laminate in-plane stiffness matrices in order to achieve the required mechanical response (the deviatoric part of the shear stiffness matrices [$\widehat{A}^*$], [$\widehat{D}^*$] and [$\widehat{F}^*$] being directly linked to the anisotropic terms of their in-plane counterparts). Moreover, as it clearly appears from Eq. (27), the ratio between the deviatoric part of matrices [$\widehat{A}^*$], [$\widehat{D}^*$] and [$\widehat{F}^*$] and the anisotropic term $R_{1M^*}e^{i2\Phi_{1M^*}}$ of their in-plane counterparts is constant once the material of the constitutive layer is chosen: such a ratio does not depend upon the layers orientations and positions, rather it solely varies with the material properties of the constitutive layer. Finally, due to the relationships (27) if one of the matrices [A*], [D*] and [F*] is characterised by a square symmetric behaviour (i.e. $R_{1M^*} = 0$) the corresponding transverse shear stiffness matrix will exhibit an isotropic behaviour (the deviatoric part of the matrix becomes null, i.e. $R_{\widehat{M}^*} = 0$).

3.2. Definition of some new classes of laminates

When looking at Eqs. (18)–(27) one can notice that the laminate elastic behaviour is governed, at the macro-scale, by an overall number of 39 polar parameters: six for each one of the matrices [A*], [B*], [D*], [E*], [F*] and [H*] together with the isotropic polar moduli of matrices [$\widehat{A}^*$], [$\widehat{D}^*$] and [$\widehat{F}^*$] (the deviatoric part being linked to the anisotropic part of their in-plane counterparts). In this set the isotropic moduli of [B*] and [E*] are null while those of the remaining matrices are equal to the isotropic moduli of the lamina reduced stiffness matrices. The only polar parameters which depend upon the geometrical characteristics of the stack are the anisotropic moduli R_{0M^*} and R_{1M^*} as well as the related polar angles Φ_{0M^*} and Φ_{1M^*} of the laminate in-plane stiffness matrices for an overall number of 24 polar parameters on which the designer can intervene to get the desired mechanical response at the macro-scale.

Technical constants		Polar parameters of [Q]		Polar parameters of [$\widehat{Q}$]	
E_1	161000 MPa	T_0	23793.3868 MPa	T	5095.4545 MPa
E_2	9000 MPa	T_1	21917.8249 MPa	R	1004.5454 MPa
G_{12}	6100 MPa	R_0	17693.3868 MPa	Φ	90 deg
ν_{12}	0.26	R_1	19072.0711 MPa		
ν_{23}	0.1	Φ_0	0 deg		
		Φ_1	0 deg		
Density and thickness					
ρ	1.58×10^{-6} kg/mm^3				
t_{ply}	0.125 mm				

Genetic parameters	
N_{pop}	1
N_{ind}	500
N_{gen}	500
p_{cross}	0.85
p_{mut}	$1/N_{\text{ind}}$
Selection	Roulette-wheel
Elitism	Active

Case n.	Solution n.	Stacking sequence	Residual
1	1	[72/14/−63/−80/−48/70/3/37/−25/ −10/11/77/−19/62/11/−54/−88/−54/ 19/77]	6.0009×10^{-6}
	2	[−1/−14/88/51/−73/−88/−32/24/20/ 62/−49/22/−44/−81/72/−9/−75/73/16/ −15]	4.8924×10^{-5}
2	1	[−31/35/32/−31/−32/30/35/−70/−21/ −38/−70/59/43/−31/−34/21/36/−29/ −38/31]	1.7476×10^{-4}
	2	[35/−43/−25/−27/23/32/56/−40/30/56/ 41/−31/−37/12/−41/−38/33/47/25/ −31]	1.9490×10^{-4}
3	1	[78/−51/−2/10/41/89/−44/2/42/−82/ 71/−53/−19/17/−8/−87/87/62/−46/11]	6.6123×10^{-4}
	2	[−3/−75/69/90/−11/5/26/−74/6/−10/ 80/86/83/−66/−5/6/−89/−19/18/88]	8.1274×10^{-4}

The set of polar parameters to be designed can be further reduced by generalising to the case of the TSDT the concept of a *quasi-homogeneous* laminate initially introduced by Vannucci and Verchery [21]. To this purpose let us introduce the laminate homogeneity matrices $[C_1^*]$, $[C_2^*]$ and $[C_3^*]$, which are defined as:

$$[C_1^*] = [A^*] - [D^*], \quad [C_2^*] = [D^*] - [F^*], \quad [C_3^*] = [F^*] - [H^*]. \quad (28)$$

In the framework of the TSDT the following definitions apply:

1. a laminate is defined *fully uncoupled* if and only if

$$[B^*] = [E^*] = [O]; \quad (29)$$

2. a laminate is said *homogeneous in bending* if and only if

$$[C_2^*] = [C_3^*] = [O]; \quad (30)$$

3. a laminate is said *homogeneous in membrane and bending* if and only if

$$[C_1^*] = [C_2^*] = [C_3^*] = [O]; \quad (31)$$

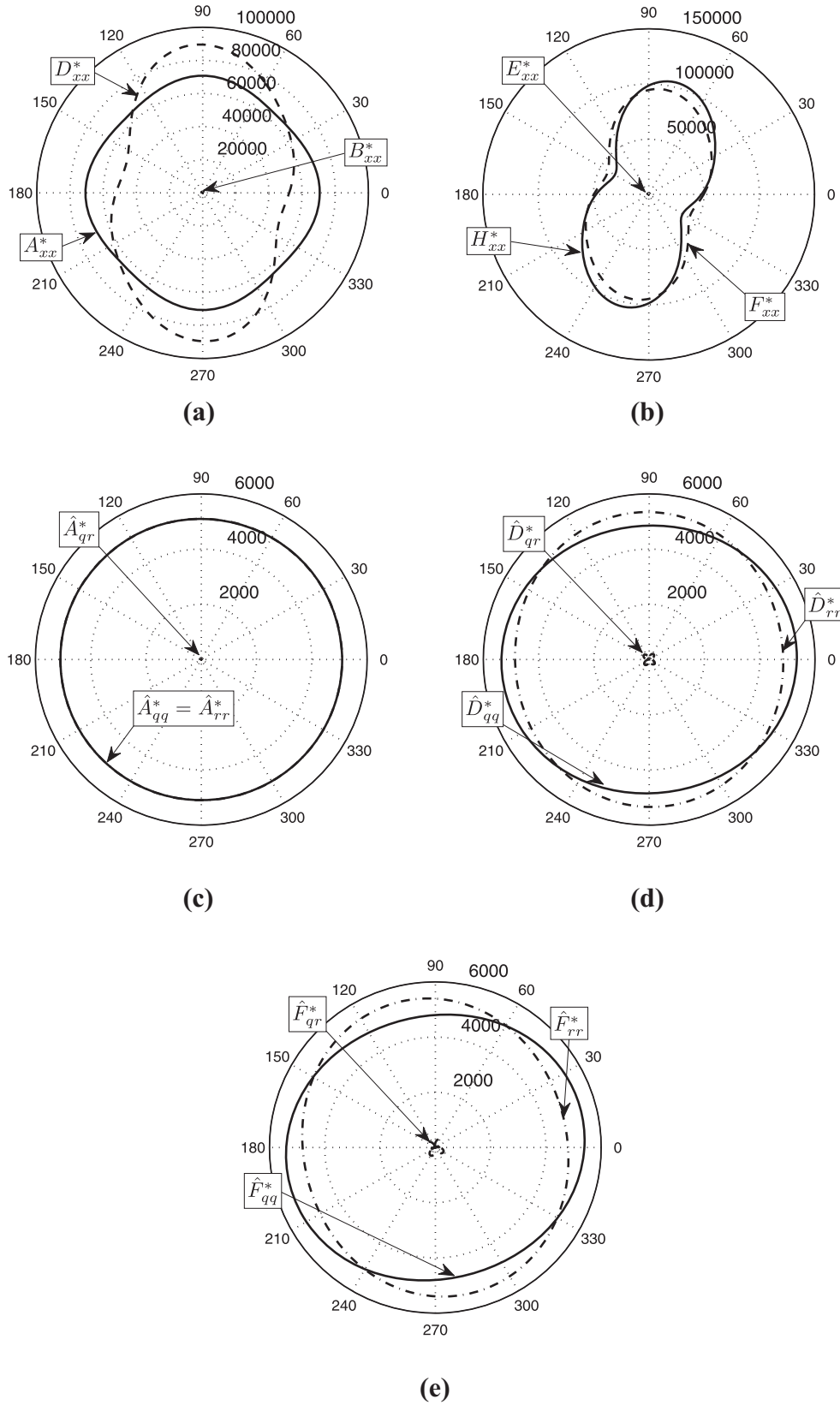4. a laminate is defined *fully uncoupled and homogeneous* if and only if

$$[B^*] = [E^*] = [C_1^*] = [C_2^*] = [C_3^*] = [O]. \quad (32)$$

It is noteworthy that, since the deviatoric part of the laminate transverse shear stiffness matrices depends upon the anisotropic one of their in-plane counterparts, Eqs. (30) and (31) imply that the resulting laminate will be homogeneous also in terms of its shear stiffness properties.

As a conclusive remark, it can be noticed that when the elastic uncoupling condition is met, the laminate mechanical response is governed by a set of 16 polar parameters, whilst if the laminate is fully uncoupled and homogeneous the number of polar parameters whereon the designer can act reduces from 24 to four, i.e. the anisotropic polar moduli and the related polar angles of matrix $[A^*]$. This last class of laminates is rather fundamental for design purposes: despite the kinematic model on which the TSDT relies is "richer" and more general than that of the CLT, in this case the number of independent tensor invariants characterising the mechanical response of the laminate and on which the designer can intervene is the same as the case of the CLT.

4. Elastic symmetries of the laminate: the polar approach in the framework of the TSDT

In this section, the problem of designing the laminate elastic symmetries is generalised to the theoretical framework of the TSDT. This problem was initially introduced by Vannucci [22] for

Polar parameters	[A^*]	[B^*]	[D^*]	[E^*]	[F^*]	[H^*]
In-plane elastic behaviour						
T_0 [MPa]	23793.3868	0	23793.3868	0	23793.3868	23793.3868
T_1 [MPa]	21917.8249	0	21917.8249	0	21917.8249	21917.8249
R_0 [MPa]	3003.1984	51.76233	4107.1180	79.6684	4586.8578	5700.1396
R_1 [MPa]	13.5942	31.1562	4739.5619	36.0186	6416.5852	8244.9550
Φ_0 [deg]	0	N.D.	9	N.D.	3	−4
Φ_1 [deg]	N.D.	N.D.	85	N.D.	75	70

Polar parameters	[$\widehat{A}^*$]		[$\widehat{D}^*$]		[$\widehat{F}^*$]	
Out-of-plane elastic behaviour						
T [MPa]	5095.4545		5095.4545		5095.4545	
R [MPa]	0.7160		249.6376		337.9681	
Φ [deg]	N.D.		5		15	

Fig. 2. Polar diagrams of the laminate stiffness matrices for the best solution of case 1. First Cartesian component of (a) [A*], [B*], [D*] and (b) [E*], [F*], [H*]; the three Cartesian components of (c) [Â*], (d) [D̂*] and (e) [F̂*].

the case of the CLT, later it has been modified and extended to the case of laminates with variable number of plies [19] and to the conceptual framework of the FSDT [20].

As widely discussed in [22], this problem can be stated as an unconstrained minimisation problem in the space of the laminate polar parameters. The goal of this problem is to find at least one

stacking sequence meeting the elastic requirements for the multi-layer plate (in terms of the elastic symmetries of the different stiffness matrices governing the laminate behaviour) provided by the problem at hand. In the context of the TSDT this unconstrained minimisation problem can be stated as:

$$\min_{\{\delta_1,\dots,\delta_n\}} \Psi(\delta_1,\dots,\delta_n) = \{f\}^T[W]\{f\}, \tag{33}$$

where Ψ is the overall objective function expressing the laminate elastic response and δ_k is the orientation angle of the kth layer ($k = 1,\dots n$). $\{f\}$ is the vector of the partial objective functions (each one linked to a particular elastic symmetry) while $[W]$ is a positive semi-definite diagonal matrix of weights whose terms can be equal to either zero or one (depending on the considered combination of elastic symmetries). The components of the vector $\{f\}$ as well as the related physical meaning are listed here below:

- $f_1 = \frac{\|[B^*]\|}{\|[Q]\|}$ is the membrane/bending uncoupling condition;
- $f_2 = \frac{\|[E^*]\|}{\|[Q]\|}$ is the higher-order membrane/bending uncoupling condition;
- $f_{2+j} = \frac{\|[C_j^*]\|}{\|[Q]\|}$ with $j = 1, 2, 3$ are the homogeneity conditions;
- $f_{5+i} = \frac{\Phi_{0M^*} - \Phi_{1M^*} - K_{M^*}\pi/4}{\pi/4}$, with $K_{M^*} = 0, 1$, is the ordinary orthotropy condition for the generic homogenised in-plane stiffness matrix $[M^*]$ of the laminate;
- $f_{9+i} = \frac{R_{0M^*}}{R_0}$ is the R_0-orthotropy condition for $[M^*]$;
- $f_{13+i} = \frac{R_{1M^*}}{R_1}$ is the square symmetry condition for $[M^*]$;
- $f_{17+i} = \frac{\sqrt{R_{0M^*}^2 + 4R_{1M^*}^2}}{\sqrt{R_0^2 + 4R_1^2}}$ is the isotropy condition for $[M^*]$;
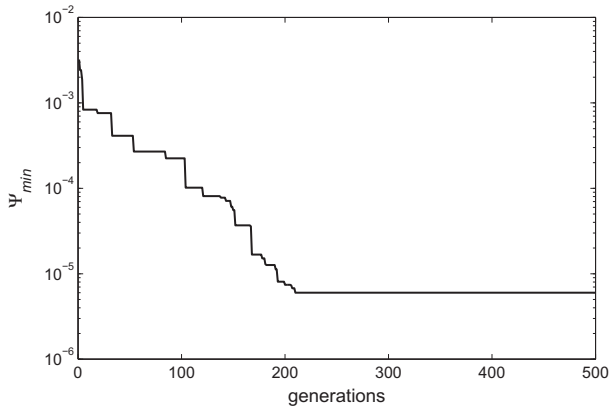

Fig. 3. Best values of the objective function along generations, case 1.

- $f_{21+i} = \frac{R_{0M^*} - \widehat{R}_{0M^*}}{\widehat{R}_{0M^*}}$ represents a condition on the value of the first anisotropic modulus of $[M^*]$ which can be used in the cases of ordinary orthotropy or square symmetry (but not in the cases of both R_0-orthotropy and isotropy);
- $f_{25+i} = \frac{R_{1M^*} - \widehat{R}_{1M^*}}{\widehat{R}_{1M^*}}$ represents a condition on the value of the second anisotropic modulus of $[M^*]$ which can be used in the cases of ordinary orthotropy or R_0-orthotropy (but not in the cases of both square symmetry and isotropy);
- $f_{29+i} = \frac{\Phi_{0M^*} - \widehat{\Phi}_{0M^*}}{\pi/4}$ represents a condition on the value of the orientation of the main orthotropy axis of $[M^*]$ which can be used in the case of square symmetry (but not in the cases of ordinary orthotropy, R_0-orthotropy and isotropy);
- $f_{33+i} = \frac{\Phi_{1M^*} - \widehat{\Phi}_{1M^*}}{\pi/4}$ represents a condition on the value of the orientation of the main orthotropy axis of $[M^*]$ which can be used in the cases of ordinary orthotropy or R_0-orthotropy (but not in the cases of both square symmetry and isotropy);

In the previous formulae $[M^*] = [A^*], [D^*], [F^*], [H^*]$ when $i = 1, 2, 3, 4$, respectively: the considered elastic condition can be imposed on each one of the homogenised in-plane stiffness matrices of the laminate, depending on the elastic requirements provided by the problem at hand.

It can be noticed that all of the components of the vector $\{f\}$ are expressed in terms of the polar parameters of the laminate homogenised in-plane stiffness matrices and that they have been normalised with the corresponding counterparts of the matrix $[Q]$. Moreover, the expression of the matrix norm used for the first five partial functions is that proposed by Kandil and Verchery [27]:

$$\|[Q]\| = \sqrt{T_0^2 + 2T_1^2 + R_0^2 + 4R_1^2}, \tag{34}$$

an analogous relationship applies for matrices $[B^*], [E^*], [C_1^*], [C_2^*]$ and $[C_3^*]$. Of course, the terms belonging to the diagonal of the weight matrix $[W]$ cannot be all different from zero at the same time: for instance it is not possible to design a laminate which is simultaneously orthotropic and isotropic in membrane, or a laminate which is homogeneous in bending with $[D^*]$ showing an orthotropic response while $[F^*]$ and $[H^*]$ having an isotropic behaviour (indeed if the laminate is homogeneous in bending it is characterised by the same elastic behaviour for $[D^*], [F^*]$ and $[H^*]$), etc. Therefore a particular care must be taken in tuning the terms of the weight matrix.

As a conclusive remark it is noteworthy that the objective function Ψ is a dimensionless, positive semi-definite convex function in the space of laminate polar parameters whose absolute minima are known a priori since they are the zeroes of this function. On the

Table 5
Laminate polar parameters for the best stacking sequence of case 2 (N.D. = not defined, i.e. meaningless for the considered combination of laminate elastic symmetries).

Polar parameters	$[A^*]$	$[B^*]$	$[D^*]$	$[E^*]$	$[F^*]$	$[H^*]$
In-plane elastic behaviour						
T_0 [MPa]	23793.3868	0	23793.3868	0	23793.3868	23793.3868
T_1 [MPa]	21917.8249	0	21917.8249	0	21917.8249	21917.8249
R_0 [MPa]	9047.8225	103.2427	10636.4139	438.0223	11013.4808	11103.5897
R_1 [MPa]	5078.2024	16.6008	7845.7886	151.2313	8020.8991	8034.9545
Φ_0 [deg]	−45	N.D.	45	N.D.	45	45
Φ_1 [deg]	−6	N.D.	0	N.D.	0	0
Polar parameters	$[\widehat{A}^*]$		$[\widehat{D}^*]$		$[\widehat{F}^*]$	
Out-of-plane elastic behaviour						
T [MPa]	5095.4545		5095.4545		5095.4545	
R [MPa]	267.4741		413.2457		422.4689	
Φ [deg]	−84		90		90	
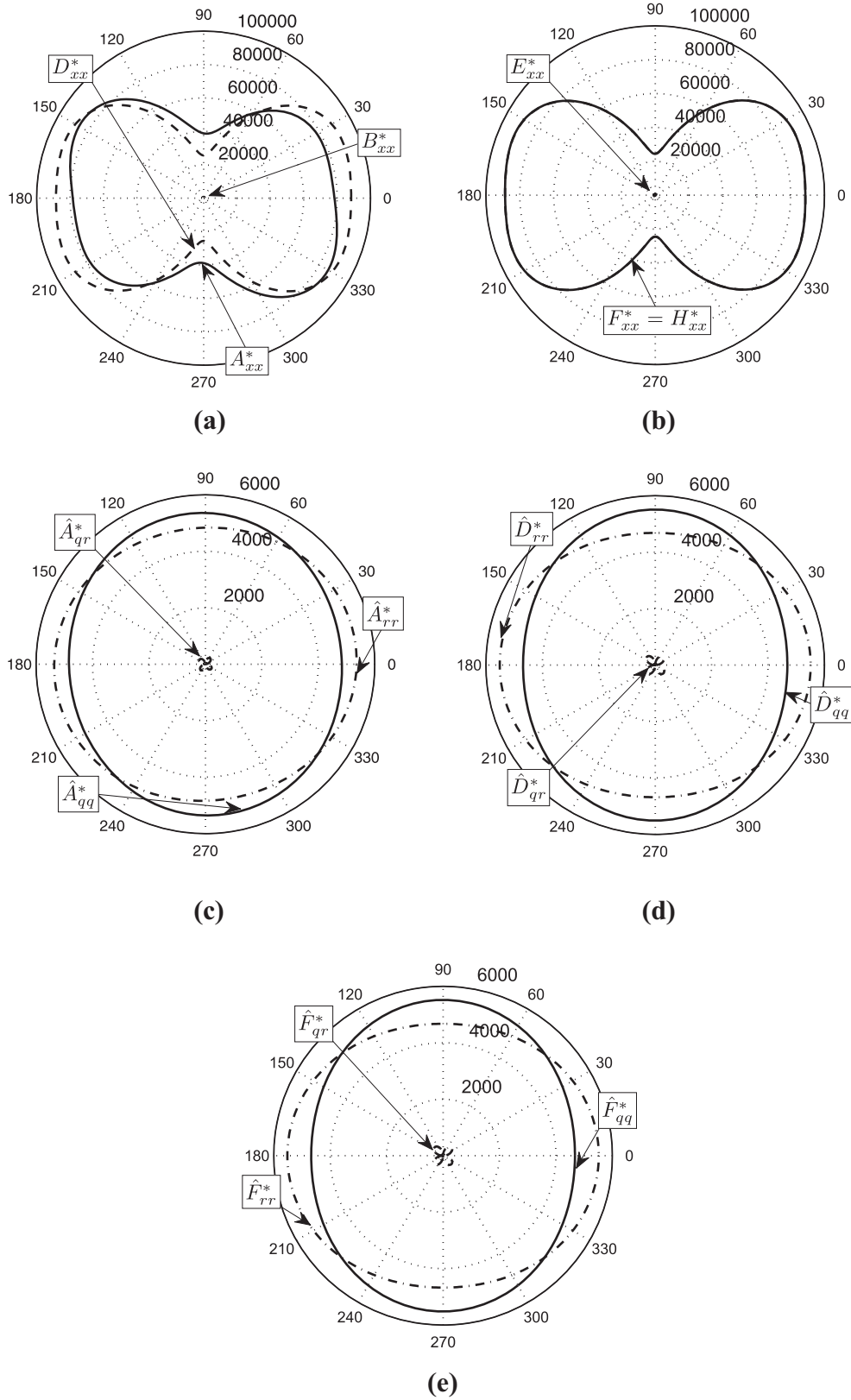
Fig. 4. Polar diagrams of the laminate stiffness matrices for the best solution of case 2. First Cartesian component of (a) [A*], [B*], [D*] and (b) [E*], [F*], [H*]; the three Cartesian components of (c) [Â*], (d) [D̂*] and (e) [F̂*].

other hand Ψ is a highly non-convex non-linear function in the space of plies orientation angles, i.e. the true design variables of problem (33), because the laminate polar parameters depend upon circular functions of these angles, see Eqs. (18)–(23). For more details about the nature of this problem the reader is addressed to [19,25].
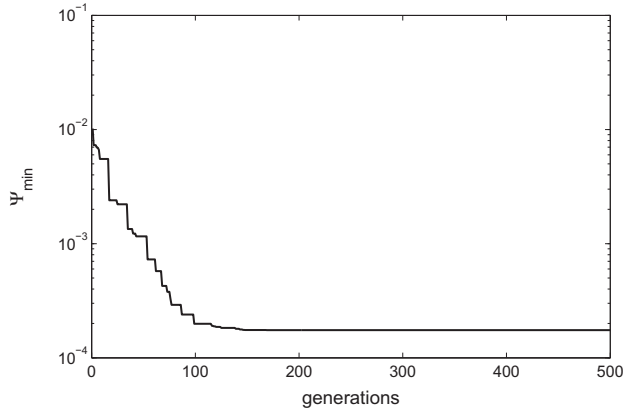
Fig. 5. Best values of the objective function along generations, case 2.

5. Studied cases and results

In this section some meaningful numerical examples concerning the problem of designing the laminate elastic behaviour are illustrated in order to numerically prove the existence of the new classes of laminates presented in Section 3. Moreover, such examples show on one hand the effectiveness of using the polar approach in the framework of the TSDT, while on the other hand they show the effectiveness of non-conventional lay-ups which satisfy (for each considered case) the prescribed set of elastic requirements imposed by the problem. In particular, the problem of designing the laminate elastic symmetries is formulated and solved in the following cases:

- fully uncoupled laminate with square symmetric membrane behaviour;
- fully uncoupled laminate homogeneous in bending with an orthotropic behaviour;
- fully uncoupled and homogeneous laminate with a square symmetric behaviour.

Since the elastic behaviour of the laminate depends upon the elastic properties of the constitutive lamina, the results must refer to a given material. In the case of the numerical examples illustrated in this section a transverse isotropic unidirectional carbon/epoxy ply has been chosen, whose material properties are listed in Table 1. In addition the number of layers n composing the laminated plate is fixed equal to 20.

Due to the nature of the optimisation problem of Eq. (33), i.e. a highly non-convex unconstrained minimisation problem in the space of the layers orientations, a genetic algorithm, BIANCA [24,25,28], has been employed to find a solution. In this case, each individual has a genotype composed of n chromosomes, i.e. one for each ply, characterised by a single gene coding the layer orientation. It must be pointed out that the orientation angle of each lamina can get all the values in the range $[-89°, 90°]$ with a discretisation step of $1°$. Such a discretisation step has been chosen in order to prove that laminates with given elastic properties can be easily obtained by abandoning the well-known conventional rules for tailoring the laminate stack (e.g. symmetric-balanced stacks) which extremely shrink the search space for the problem at hand. Therefore, the true advantages in using non-conventional staking sequences are at least two: on one hand when using such a discretisation step for the plies orientations it is possible to explore the overall design space of problem (33), while on the other hand the polar-genetic approach leads to find very general stacks (nor symmetric neither balanced) that fully meet the elastic properties with a fewer number of plies (hence lighter) than the standard ones. For more details about these aspects the reader is addressed to [19,25].

Finally, regarding the value of the genetic parameters for the GA BIANCA they are listed in Table 2. For more details on the numerical techniques developed within the BIANCA code and the meaning of the values of the different parameters tuning the GA the reader is addressed to [25,28].

5.1. Case 1: fully uncoupled laminate with square symmetric membrane behaviour

The aim of this first case is the design of a fully uncoupled laminate with the membrane stiffness matrix $[A^*]$ characterised by a square symmetric behaviour with a prescribed orientation of the main orthotropy axis, i.e. in this case $\widehat{\Phi}_{0A^*} = 0°$. Moreover, when imposing this kind of symmetry on matrix $[A^*]$ the designer can automatically obtain an isotropic behaviour for matrix $[\widehat{A}^*]$, as a consequence of Eq. (27). In this case, the expression of the overall objective function Ψ of Eq. (33) can be obtained by setting $W_{11} = W_{22} = W_{1414} = W_{3030} = 1$ and the rest of the terms of the diagonal matrix $[W]$ equal to zero:

$$\Psi = \left(\frac{\|[B^*]\|}{\|[Q]\|}\right)^2 + \left(\frac{\|[E^*]\|}{\|[Q]\|}\right)^2 + \left(\frac{R_{1A^*}}{R_1}\right)^2 + \left(\frac{\Phi_{0A^*} - \widehat{\Phi}_{0A^*}}{\pi/4}\right)^2. \quad (35)$$

Table 3 shows two examples of laminate stacking sequences satisfying the criteria of Eq. (35). The residual in the last column is the value of the objective function Ψ for each solution (recall that exact solutions correspond to zeros of the objective function, see [22] for more details). It can be noticed that the optimal stacking sequences are really general: they are nor symmetric neither balanced and they fully meet the elastic symmetry requirements imposed on the laminate through Eq. (35) with only 20 plies.

Table 4 lists the value of the laminate polar parameters for the best stacking sequence (solution n. 1) of Table 3, while Fig. 2

Table 6
Laminate polar parameters for the best stacking sequence of case 3 (N.D. = not defined, i.e. meaningless for the considered combination of laminate elastic symmetries).

Polar parameters	$[A^*]$	$[B^*]$	$[D^*]$	$[E^*]$	$[F^*]$	$[H^*]$
In-plane elastic behaviour						
T_0 [MPa]	23793.3868	0	23793.3868	0	23793.3868	23793.3868
T_1 [MPa]	21917.8249	0	21917.8249	0	21917.8249	21917.8249
R_0 [MPa]	3160.6773	403.7186	3449.9296	410.40234	3176.0704	3497.94823
R_1 [MPa]	40.9970	164.6285	71.8450	503.3974	231.5776	237.4147
Φ_0 [deg]	0	N.D.	0	N.D.	0	0
Φ_1 [deg]	N.D.	N.D.	N.D.	N.D.	N.D.	N.D.
Polar parameters	$[\widehat{A}^*]$		$[\widehat{D}^*]$		$[\widehat{F}^*]$	
Out-of-plane elastic behaviour						
T [MPa]	5095.4545		5095.4545		5095.4545	
R [MPa]	2.1594		3.7842		12.1974	
Φ [deg]	N.D.		N.D.		N.D.	
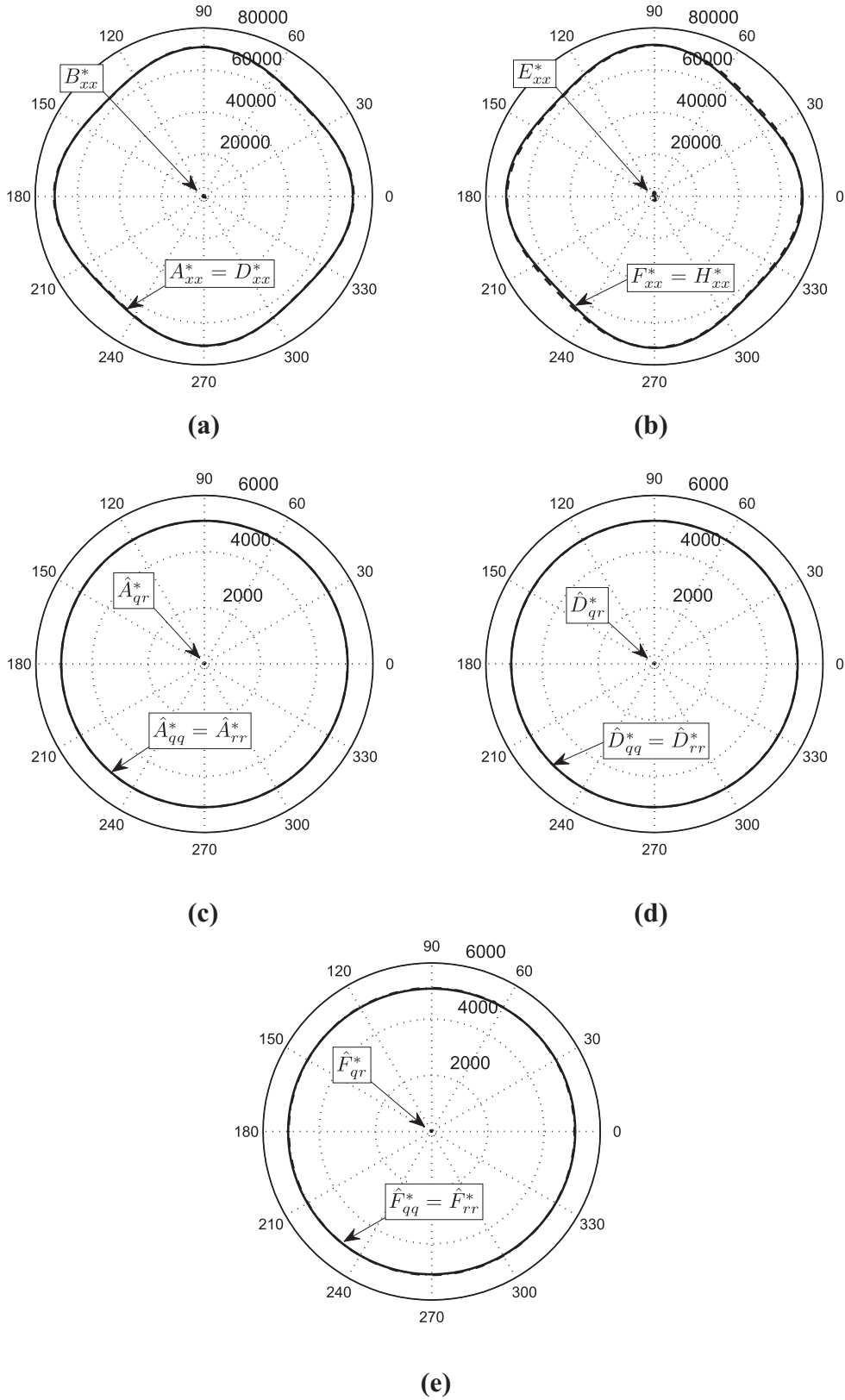
Fig. 6. Polar diagrams of the laminate stiffness matrices for the best solution of case 3. First Cartesian component of (a) [A*], [B*], [D*] and (b) [E*], [F*], [H*]; the three Cartesian components of (c) [Â*], (d) [D̂*] and (e) [F̂*].

illustrates the related polar diagrams for the first Cartesian component of matrices [A*], [B*], [D*], [E*], [F*], [H*] and the three Cartesian components of matrices [Â*], [D̂*] and [F̂*]. One can notice that,

according to the theoretical result of Eq. (27), the laminate is characterised both by a square symmetric membrane stiffness behaviour (whose main orthotropic axis is oriented at 0°, see Table 4)

and by an isotropic elastic response for matrix $[\widehat{A}^*]$. In addition the laminate is practically uncoupled (B_{xx}^* and E_{xx}^* reduce to a small point in the centre of the plot) while it is completely anisotropic in bending because no elastic requirements have been imposed on $[D^*]$, $[F^*]$ and $[H^*]$. Moreover, as a consequence of the anisotropy of matrices $[D^*]$ and $[F^*]$, the higher-order transverse shear matrices $[\widehat{D}^*]$ and $[\widehat{F}^*]$ show a general orthotropic behaviour (the deviatoric modulus of these matrices does not vanish, see Table 4). It is noteworthy that such results have been found with very general stacks composed of a few number of plies: it is really difficult (if not impossible) to obtain the same laminate mechanical response with standard multilayer plates, i.e. plates characterised by a symmetric, balanced lay-up.

As a final remark, Fig. 3 shows the variation of the value of the objective function of the best solution (of Table 3) along generations for problem (33) for this first case. One can easily see that the optimum solution has been found only after 210 generations. Since the problem is highly non-convex, at the end of the genetic calculation it is possible to find within the population not only the best solution but also some fitting quasi-optimal solutions like solution n. 2 of Table 3: the presence of such solutions (whereof solution n. 2 is only an example among the others composing the final population) can be effectively exploited by the designer which wants to deeply investigate their mechanical response with respect to different design criteria (e.g. buckling, natural frequencies, etc.).

5.2. Case 2: fully uncoupled laminate homogeneous in bending with an orthotropic behaviour

For this second case the goal consists in designing a fully uncoupled laminate which must be simultaneously homogeneous in bending and characterised by a flexural orthotropic behaviour (with $K_{D^*} = 1$) with a prescribed direction of the main orthotropy axis (in this case $\widehat{\Phi}_{1D^*} = 0°$).

The expression of the overall objective function Ψ of Eq. (33) can be obtained by setting $W_{11} = W_{22} = W_{44} = W_{55} = W_{77} = W_{3535} = 1$ and the rest of the terms of the diagonal matrix $[W]$ equal to zero:

$$\Psi = \left(\frac{\|[B^*]\|}{\|[Q]\|}\right)^2 + \left(\frac{\|[E^*]\|}{\|[Q]\|}\right)^2 + \left(\frac{\|[C_2^*]\|}{\|[Q]\|}\right)^2 + \left(\frac{\|[C_3^*]\|}{\|[Q]\|}\right)^2$$
$$+ \left(\frac{\Phi_{0D^*} - \Phi_{1D^*} - \pi/4}{\pi/4}\right)^2 + \left(\frac{\Phi_{1D^*} - \widehat{\Phi}_{1D^*}}{\pi/4}\right)^2. \quad (36)$$

Two examples of laminate stacking sequences satisfying the criteria of Eq. (36) are listed in Table 3. Table 5 lists the value of the laminate polar parameters for the best stacking sequence (solution n. 1) of Table 3, while Fig. 4 illustrates the related polar diagrams of the stiffness matrices of the laminate. One can notice that the laminate is orhtotropic in bending because the shape of the polar diagrams for the matrices $[D^*]$, $[F^*]$ and $[H^*]$ shows two axes of orthogonal symmetry with the main orthotropy axis oriented at $0°$. In addition, the laminate is really homogeneous in bending because the previous diagrams are superposed: this superposition also applies for the polar diagrams of the components of matrices $[\widehat{D}^*]$ and $[\widehat{F}^*]$. Furthermore, the laminate is fully uncoupled (B_{xx}^* and E_{xx}^* reduce to a small point in the centre of the plot) while it is completely anisotropic in terms of the elastic response of the matrix $[A^*]$.

Finally, Fig. 5 shows the variation of the value of the objective function for the best solution (of Table 3) along generations for problem (33) for this second case. It can be noticed that the
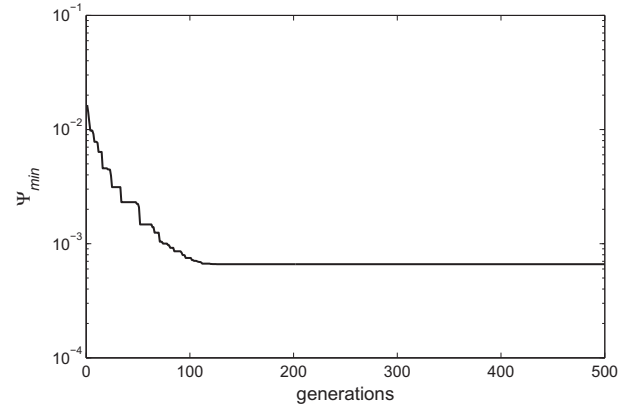

Fig. 7. Best values of the objective function along generations, case 3.

optimum solution has been found after 185 generations. For the rest, the considerations already done for case 1 can be repeated here.

5.3. Case 3: fully uncoupled and homogeneous laminate with a square symmetric behaviour

The aim of this example is the design of a fully uncoupled homogeneous laminate with an overall square symmetric elastic behaviour (both in extension and bending) and with the main axis of symmetry oriented at $\widehat{\Phi}_{0A^*} = 0°$.

The expression of the overall objective function Ψ of Eq. (33) can be obtained by setting $W_{11} = W_{22} = W_{33} = W_{44} = W_{55} = W_{1414} = W_{3030} = 1$ and the rest of the terms of the diagonal matrix $[W]$ equal to zero:

$$\Psi = \left(\frac{\|[B^*]\|}{\|[Q]\|}\right)^2 + \left(\frac{\|[E^*]\|}{\|[Q]\|}\right)^2 + \left(\frac{\|[C_1^*]\|}{\|[Q]\|}\right)^2 + \left(\frac{\|[C_2^*]\|}{\|[Q]\|}\right)^2$$
$$+ \left(\frac{\|[C_3^*]\|}{\|[Q]\|}\right)^2 + \left(\frac{R_{1A^*}}{R_1}\right)^2 + \left(\frac{\Phi_{0A^*} - \widehat{\Phi}_{0A^*}}{\pi/4}\right)^2. \quad (37)$$

Two examples of laminate stacking sequences satisfying the criteria of Eq. (37) are listed in Table 3: also in this case the optimal stacks are very general stacks. Table 6 lists the value of the laminate polar parameters for the best stacking sequence (solution n. 1) of Table 3, while Fig. 6 illustrates the related polar diagrams for all the stiffness matrices of the multilayer plate. It is noteworthy that, due to the theoretical result of Eq. (27), the laminate is characterised both by an overall square symmetric elastic response (for each one of the in-plane stiffness matrices) and by a global isotropic out-of-plane shear behaviour. Moreover the laminate is fully uncoupled and homogeneous (the polar diagrams of matrices $[A^*]$, $[D^*]$, $[F^*]$ and $[H^*]$ are in fact superposed). Finally, the main axis of symmetry for every in-plane stiffness matrix is oriented at $0°$.

As a final remark of this section, Fig. 7 shows the variation of the value of the objective function for the best solution (of Table 3) along generations for problem (33) for this last case: the optimum solution has been found after about 105 generations. For the rest, the considerations already done for cases 1 and 2 can be repeated here.

6. Conclusions

In this work the Verchery's polar method for representing plane tensors has been employed within the conceptual framework of the classical Reddy's Third-order Shear Deformation

Theory of laminates. The following major results were analytically derived.

1. A mathematical formulation based upon tensor invariants has been proposed for all the laminate stiffness matrices.
2. As a consequence of the application of the polar formalism to the higher-order in-plane and transverse shear stiffness matrices a generalisation of the concept of *quasi-homogeneous* laminates has been proposed together with the definition of some new classes of laminates.

3. The elastic symmetries of the laminate out-of-plane shear stiffness matrices (basic and higher-order terms) depend upon those of their in-plane counterparts: in particular, the isotropic behaviour of the laminate shear stiffness matrices is closely related to the square symmetric behaviour of their in-plane counterparts.
4. For a special class of laminates, i.e. for *fully uncoupled and homogeneous* laminates, the number of independent tensor invariants characterising the mechanical response of the laminate remains unchanged when passing from the context of the CLT to that of the TSDT.
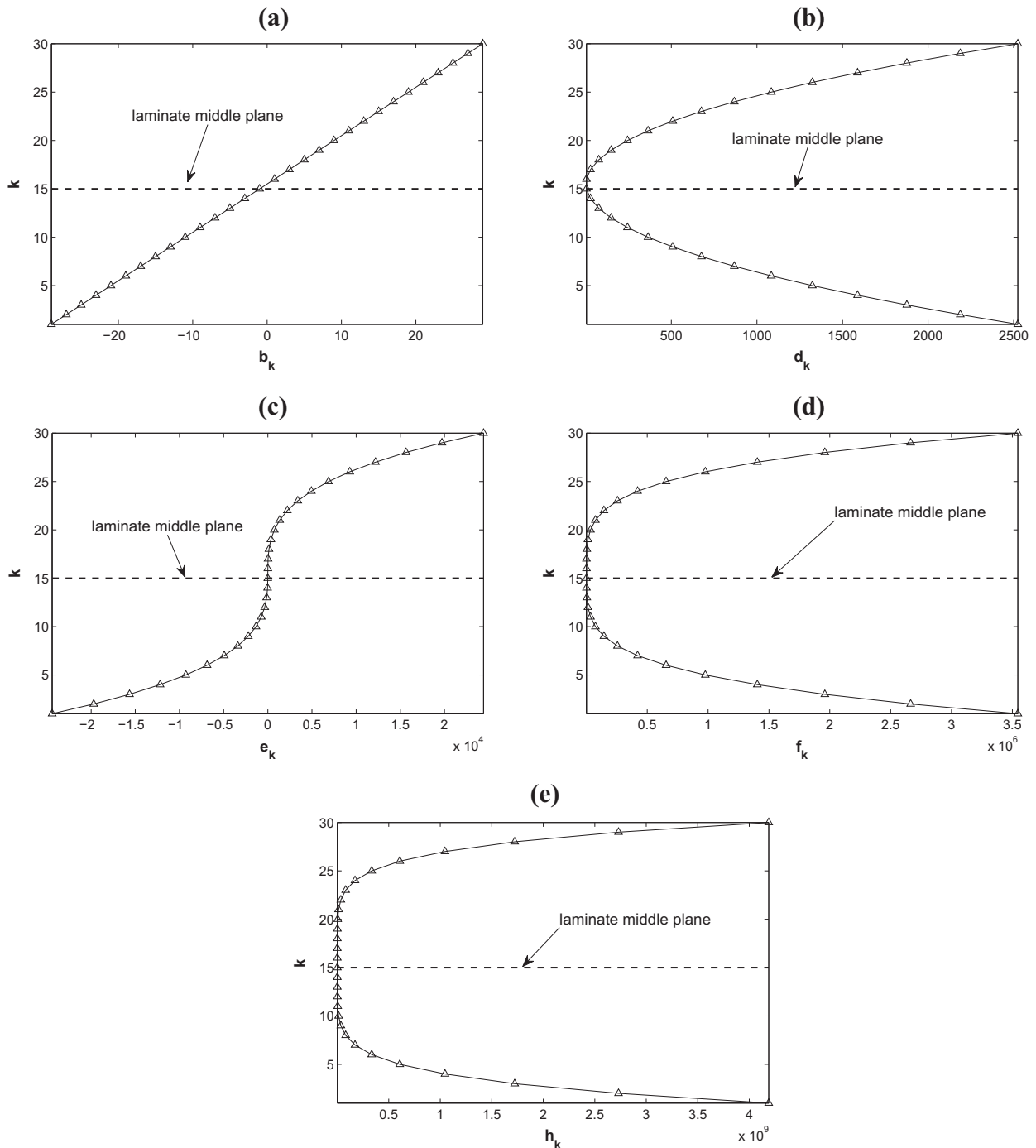
Fig. A.1. Variation of the laminate stiffness coefficients (a) b_k, (b) d_k, (c) e_k, (d) f_k and (e) h_k vs. the layer position k (for the case $n = 30$).

5. The unified formulation of the problem of designing the laminate elastic symmetries has been modified and extended to the context of the TSDT.

To the best of the author's knowledge, this is the first time that a mathematical formulation based upon tensor invariants (namely the polar method) has been applied to the conceptual framework of the TSDT. The mechanical response of the laminated plate is represented by means of the polar formalism that offers several advantages: (a) the polar invariants are directly linked to the tensor elastic symmetries, (b) the polar method allows for eliminating from the procedure redundant mechanical properties and (c) it allows for easily expressing the change of reference frame.

The effectiveness of the proposed approach has been proved both analytically and numerically by means of some new and meaningful numerical examples. The numerical results presented in this work (which have been found in the context of the polar-genetic approach) show that when the well-known hypotheses and rules for tailoring laminates are abandoned (i.e. when using symmetric, balanced stacks and when considering a small set of layer orientations shrunk to the values $0°, \pm45°$ and $90°$) it is possible to design laminates with enhanced mechanical properties, very difficult (if not impossible) to be obtained otherwise.

Finally, it is opinion of the author that the polar-genetic approach can be extended also to the theoretical framework of more accurate theories such as higher-order theories coupled with equivalent single layer kinematic models: research is ongoing on these topics.

Appendix A. Analytical expression of the coefficients of the laminate stiffness matrices in the case of identical plies

In order to determine the analytical expression of the coefficients that intervene in the formulation of the stiffness matrices of a laminate composed of identical plies, see Eqs. (13) and (14), the first step is the definition of the thickness coordinate for both the bottom and the top faces of the kth elementary layer. Since the laminate is composed of n identical plies and being h its overall thickness, the expressions of z_{k-1} and z_k can be stated as (see Fig. 1):

$$z_{k-1} = -\frac{h}{2} + (k-1)\frac{h}{n}, \quad z_k = -\frac{h}{2} + k\frac{h}{n}, \quad k = 1, \ldots, n. \quad (A.1)$$

To derive the expression of the laminate characteristic coefficients b_k, d_k, e_k, f_k and h_k it suffices to substitute Eq. (A.1) in Eq. (11). Whilst the calculation of coefficients b_k and d_k is trivial and represents a classical result that can be found in literature [29], on the other hand the determination of the analytical expression of coefficients e_k, f_k and h_k is quite hard and needs the support of a symbolic-based computational software, such as Mathematica®. By means of this language it is possible to derive the following relationships:

$$
\begin{aligned}
b_k &= 2k - n - 1, \\
d_k &= 12k^2 - 12k(n+1) + 3n^2 + 6n + 4, \\
e_k &= 8k^3 - 12k^2(n+1) + 2k(3n^2 + 6n + 4) - (n^3 + 3n^2 + 4n + 2), \\
f_k &= 80k^4 - 160k^3(n+1) + 40k^2(3n^2 + 6n + 4) \\
&\quad -40k(n^3 + 3n^2 + 4n + 2) \\
&\quad +5n^4 + 20n^3 + 40n^2 + 40n + 16, \\
h_k &= 448k^6 - 1344k^5(n+1) + 560k^4(3n^2 + 6n + 4) + \\
&\quad -1120k^3(n^3 + 3n^2 + 4n + 2) + 84k^2(5n^4 + 20n^3 \\
&\quad +40n^2 + 40n + 16) + \\
&\quad -28k(3n^5 + 15n^4 + 40n^3 + 60n^2 + 48n + 16) + \\
&\quad +7n^6 + 42n^5 + 140n^4 + 280n^3 + 336n^2 + 224n + 64.
\end{aligned}
$$

$$(A.2)$$

The variation of the previous coefficients as function of the ply position k is depicted in Fig. A.1. It can be noticed that coefficients b_k and e_k have a skew-symmetric trend with respect to the laminate middle plane, whilst the rest of the coefficients show a symmetric variation. By means of the software Mathematica® one can also determine the following fundamental analytical properties characterising the laminate stiffness coefficients:

$$
\sum_{k=1}^{n} b_k = 0, \quad \sum_{k=1}^{n} d_k = n^3, \quad \sum_{k=1}^{n} e_k = 0,
$$
$$
\sum_{k=1}^{n} f_k = n^5, \quad \sum_{k=1}^{n} h_k = n^7.
$$

$$(A.3)$$

It is noteworthy that the sum of coefficients b_k and e_k is null and that this result agrees with the skew-symmetric variation of these coefficients within the laminate thickness.

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